Online Weight Balancing on the Unit Circle

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Abstract

We consider a problem as follows: Given unit weights arriving in an online manner with the total cardinality unknown, upon each arrival we decide where to place it on the unit circle in $\mathbb{R}^2$. The objective is to set the center of mass of the placed weights as close to the origin as possible. We apply competitive analysis defining the competitive difference as a performance measure. We first present an optimal strategy for placing unit weights which achieves a competitive difference of $\frac{1}{5}$. We next consider a variant in which the destination of each weight must be chosen from a set of positions that equally divide the unit circle. We give a simple strategy whose competitive difference is 0.35. Moreover, in the offline setting, several conditions for the center of mass to lie at the origin are derived.

1 Introduction

Suppose that we are given a series of points, each with unit weight, one by one with the total cardinality unknown in advance. Our task is to place the points one by one on the unit circle in $\mathbb{R}^2$ while keeping a good balance. We are not allowed to move the point any more, once it is placed. The balance is measured by the Euclidean distance between the center of mass of the placed points and the origin.

The difficulty is that we do not know how many points will arrive in total. If we guess the total cardinality somehow at the beginning, then we may try to place the points, for example, in such a way that they equally divide the unit circle. If the guess is correct, the center of mass comes to the origin. However, if the guess fails, say, if one extra point arrives, we have to place it somewhere and then lose the good balance. Also in the case of fewer points than expected, we cannot achieve the balance as planned. In this paper we consider this problem from the viewpoint of *competitive analysis*.

1.1 Our Contribution

We apply competitive analysis adopting the *competitive difference* as a criterion of competitiveness of a strategy. The competitive difference is defined as the maximum difference between the cost incurred by the strategy and the cost incurred by an optimal offline strategy that knows the total cardinality of points in advance. Our results are summarized as follows:

(a) We present a non-trivial optimal strategy whose competitive difference is $\frac{1}{5}$. This means that according to our strategy, the cost is guaranteed to be at most the optimal offline cost plus $\frac{1}{5}$.

(b) We impose the n-cyclotomic constraint on the problem that for fixed n, the destination of each point has to be chosen from $\{(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}) \mid 0 \le k \le n-1, k \in \mathbb{Z}\}$ and each position is occupied by at most one point. Depending on the parity of n, we give a simple and competitive strategy. Our strategy guarantees a competitive difference of 0.35 for odd n and $\frac{1}{3}$ for even n.

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(c) We investigate the n-cyclotomic constrained problem in the offline setting, in which the cardinality of points is informed at the beginning. Even with the information of the cardinality, it is not clear whether there is a placement of points that lets the center of mass come exactly to the origin. We reveal several conditions for the existence of such a placement.

1.2 Related Work

To the best of our knowledge, this paper seems the first to focus on the placement of weighted objects that *arrive in an online manner* in terms of the optimal placement of their center of mass. One can find many studies with similar purposes in the offline setting: Kurebe et al. [KMI07] considered the placement of weighted rectangles on $\mathbb{R}^2$ to let their center of mass approach the target position. Teramoto et al. [TAKD06] studied the insertion of points into the unit square in $\mathbb{R}^d$ in such a way that the Euclidean distance between any pair of points becomes as uniform as possible. Recently, Barba et al. [BCF$^+$13] considered the problem that given a set of weights, a closed connected region, and a target position, we are asked to place the weights on the boundary of the region so that the center of mass lies at the target.

In consistent hashing, one can think that items and caching machines are both mapped to points on the unit circle [KLL$^+$97, KSB$^+$99]. In the context of the space science, satellite constellation design for covering the Earth's sphere has been of great interest, for example [dWSS08, Uly09].

2 Problem Statement and Preliminaries

Throughout this paper a *point* denotes an individual object that is to be placed (or has been placed) on $\mathbb{R}^2$, while a *position* stands for where to place a point on $\mathbb{R}^2$. Each point has unit weight unless we specify otherwise. We sometimes identify a position on $\mathbb{R}^2$ and its xy coordinate, such as the origin $O = (0, 0)$. $\overline{AB}$ denotes the Euclidean distance between the positions A and B.

We define the *online weight balancing problem* as follows. We are given a series of points, each with unit weight, in an online manner where the points arrive one by one and the total cardinality is unknown in advance. Our task is to place each point, upon its arrival, somewhere on the unit circle in $\mathbb{R}^2$. Once a point is placed, it cannot be moved any more. The objective is to minimize the *cost* which is defined as the Euclidean distance between the center of mass of the placed points and the origin, that is, the center of the unit circle.

A *strategy* for placing points is denoted by a sequence $\boldsymbol{\theta} := (\theta_1, \theta_2, \ldots) \in S$ in the sense that it places the j-th point at $P_{\boldsymbol{\theta}}(j) := (\cos\theta_j, \sin\theta_j)$, where S is the set of feasible strategies (specified later). The reason why a strategy is denoted thus simply is that any adaptive decision based on the history of the configuration does not help in this problem. When k points have arrived so far and been placed according to the strategy $\boldsymbol{\theta}$, the center of mass of the points lies at

$$G_{\boldsymbol{\theta}}(k) := \Big(\frac{1}{k}\sum_{j=1}^{k}\cos\theta_j, \frac{1}{k}\sum_{j=1}^{k}\sin\theta_j\Big).$$

Then, the cost of the strategy $\boldsymbol{\theta}$ is written as

$$C_{\boldsymbol{\theta}}(k) := \overline{OG_{\boldsymbol{\theta}}(k)} = \sqrt{\Big(\frac{1}{k}\sum_{j=1}^{k}\cos\theta_j\Big)^2 + \Big(\frac{1}{k}\sum_{j=1}^{k}\sin\theta_j\Big)^2}.$$

On the other hand, the *optimal offline cost*, that is, one with the cardinality known to be k in advance, is

$$C_{opt}(k) := \inf\{C_{\boldsymbol{\theta}}(k) \mid \boldsymbol{\theta} \in S\}.$$

The performance of strategies for online problems is usually measured by the competitive ratio (see [BE98] for example), which would be defined as $\sup_{k\geq 1} \frac{C_{\boldsymbol{\theta}}(k)}{C_{opt}(k)}$ for our problem. However, this is inconvenient here since $C_{opt}(k) = 0$ and $C_{\boldsymbol{\theta}}(k) > 0$ happen often in the same time. We thus define and use the *competitive difference* instead. We say that the strategy $\boldsymbol{\theta}$ has a competitive difference of d if

$$C_{\boldsymbol{\theta}}(k) - C_{opt}(k) \leq d$$

holds for all $k \geq 1$. Apparently, $d \geq 0$. A smaller competitive difference means a better strategy.

In this paper we consider the online weight balancing problem under two different settings:

(A) The *basic problem.* We are allowed to place a point on an arbitrary position on the unit circle. Namely, the set of feasible strategies S is

$$\{(\theta_1, \theta_2, \ldots) \mid 0 \leq \theta_j < 2\pi \text{ for } j \geq 1\}.$$

(B) The *n-cyclotomic problem.* For fixed n, the destination of each point is chosen from $\{(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}) \mid 0 \leq k \leq n-1, k \in \mathbb{Z}\}$, that is, a set of n positions that equally divide the unit circle into n arcs. Any position should not be occupied more than once. Formally, we set S to

$$\left\{\left(\frac{2m_1\pi}{n}, \frac{2m_2\pi}{n}, \ldots, \frac{2m_n\pi}{n}\right) \;\middle|\; 0 \leq m_j \leq n-1, m_j \in \mathbb{Z}^n \text{ for } 1 \leq j \leq n; m_j \neq m_k \text{ for } 1 \leq j < k \leq n\right\}.$$

We assume in addition that the cardinality of the arriving points is at most n.

3 The Basic Problem

3.1 Optimal Online Strategy

We first show a lower bound on the competitive difference and then give a strategy whose competitive difference coincides with that value. We begin by presenting a simple lemma on the offline cost.

Lemma 1. *For the basic problem, it holds that*

$$C_{opt}(k) = \begin{cases} 1, & k = 1; \\ 0, & k \geq 2. \end{cases}$$

Proof. It is trivial that $C_{opt}(1) = 1$ since the cost is one wherever we place a single point. For $k \geq 2$, just adopt the strategy $(0, \frac{2\pi}{k}, \frac{4\pi}{k}, \ldots, \frac{2(k-1)\pi}{k})$. □

By rotational symmetry, we can assume that an optimal strategy satisfies $\theta_1 = 0$ and $0 \leq \theta_2 \leq \pi$. Let $\alpha := 2\arccos\frac{1}{5}$ (≈ 157 degrees), which is a key angle for obtaining an optimal strategy. The next lemma gives a lower bound on the competitive difference.

Lemma 2. *Any strategy for the basic problem has a competitive difference of at least $\frac{1}{5}$.*

Proof. Fix a strategy $\boldsymbol{\theta}$ arbitrarily. By rotational symmetry, we can assume that $\theta_1 = 0$ and $0 \le \theta_2 \le \pi$. We will show that the competitive difference is at least $\frac{1}{5}$ regardless of the value of θ_2.

(i) Case $0 \le \theta_2 < \alpha$. We have

$$G_{\boldsymbol{\theta}}(2) := \Big(\frac{1}{2}(1+\cos\theta_2), \frac{1}{2}\sin\theta_2\Big).$$

Since $x \mapsto \cos\frac{x}{2}$ is a decreasing function on $[0,\pi]$,

$$C_{\boldsymbol{\theta}}(2) = \frac{1}{2}\sqrt{(1+\cos\theta_2)^2 + \sin^2\theta_2} = \cos\frac{\theta_2}{2} > \cos\frac{\alpha}{2} = \frac{1}{5}.$$

On the other hand, $C_{opt}(2) = 0$ by Lemma 1. Therefore, the competitive difference is greater than $\frac{1}{5}$.

(ii) Case $\alpha \le \theta_2 < \pi$. We evaluate the cost after the third point has been placed. For ease of analysis, we square the cost:

$$\begin{aligned} C_{\boldsymbol{\theta}}(3)^2 =& \Big(\frac{1}{3}\sum_{j=1}^{3}\cos\theta_j\Big)^2 + \Big(\frac{1}{3}\sum_{j=1}^{3}\sin\theta_j\Big)^2 \\ =& \frac{2}{9}\sin\theta_2\sin\theta_3 + \frac{2}{9}\cos\theta_3\cos\theta_2 + \frac{2}{9}\cos\theta_2 + \frac{2}{9}\cos\theta_3 + \frac{1}{3}. \end{aligned}$$

Let us think of $C_{\boldsymbol{\theta}}(3)^2$ as a function of θ_3 with a fixed parameter θ_2. By differentiating $C_{\boldsymbol{\theta}}(3)^2$ with respect to θ_3, we obtain

$$\begin{aligned} \frac{\partial C_{\boldsymbol{\theta}}(3)^2}{\partial\theta_3} &= -\frac{2}{9}\sin\theta_3 + \frac{2}{9}\sin\theta_2\cos\theta_3 - \frac{2}{9}\sin\theta_3\cos\theta_2 \\ &= \frac{4}{9}\sin\Big(\frac{\theta_2}{2} - \theta_3\Big)\cos\frac{\theta_2}{2}. \end{aligned}$$

This implies that when $\theta_3 = \frac{\theta_2}{2} + \pi$, the function $C_{\boldsymbol{\theta}}(3)^2$ achieves a minimum of

$$\frac{2}{9}\sin\theta_2\sin\Big(\frac{\theta_2}{2}+\pi\Big) + \frac{2}{9}\cos\Big(\frac{\theta_2}{2}+\pi\Big)\cos\theta_2 + \frac{2}{9}\cos\theta_2 + \frac{2}{9}\cos\Big(\frac{\theta_2}{2}+\pi\Big) + \frac{1}{3} = \frac{1}{9}\Big(1 - 2\cos\frac{\theta_2}{2}\Big)^2.$$

(Geometrically speaking, the optimal position of the third point is the midpoint of the longer arc connecting $P_{\boldsymbol{\theta}}(1)$ and $P_{\boldsymbol{\theta}}(2)$.) Hence, for general θ_3, it holds that

$$C_{\boldsymbol{\theta}}(3) \ge \frac{1}{3}\Big(1 - 2\cos\frac{\theta_2}{2}\Big) = \frac{1}{3}\Big(1 - \frac{2}{5}\Big) = \frac{1}{5}.$$

Again by Lemma 1, we know $C_{opt}(3) = 0$. Therefore, the competitive difference is at least $\frac{1}{5}$. □

The strategy $\overline{\boldsymbol{\theta}}$ defined below turns out to be optimal. Note that the choice of placement for the fourth and later points is a matter of taste; any placement is acceptable as long as the resulting cost does not exceed $\frac{1}{5}$. Here we choose a placement for the fourth and later points so that the analysis is easy to handle. See Figure 1.

$$\overline{\theta}_j = \begin{cases} 0, & j = 1; \\ \alpha, & j = 2; \\ \frac{\alpha}{2} + \pi, & j \text{ is odd, } j \ge 3; \\ \frac{\alpha}{2}, & j \text{ is even, } j \ge 4. \end{cases}$$

Lemma 3. *$C_{\overline{\boldsymbol{\theta}}}(1) = 1$, and $C_{\overline{\boldsymbol{\theta}}}(k) \le \frac{1}{5}$ for all $k \ge 2$.*

Proof. $C_{\overline{\boldsymbol{\theta}}}(1) = 1$ is trivial. For ease of notation we write $P_{\overline{\boldsymbol{\theta}}}(\cdot)$ and $G_{\overline{\boldsymbol{\theta}}}(\cdot)$ simply as $P(\cdot)$ and $G(\cdot)$, respectively. Although the lemma can be proved by explicitly calculating the coordinate of $G(k)$ for general $k \ge 2$, we here give a simpler proof based on geometric arguments. Applying the strategy $\overline{\boldsymbol{\theta}}$, we calculate $G(2) = (\frac{1}{25}, \frac{2\sqrt{6}}{25})$ and $G(3) = (-\frac{1}{25}, -\frac{2\sqrt{6}}{25})$. (See Figure 1.) It is thus observed that the origin O lies on the segment $G(2)G(3)$ and $\overline{OG(2)} = \overline{OG(3)} = \frac{1}{5}$. Therefore, the proof is done if $G(k)$ lies on the segment $G(2)G(3)$ for all $k \ge 2$.

We begin by proving that every $G(k)$ is on the line $G(2)G(3)$, not necessarily on that segment. Please note that $P(3) = P(5) = P(7) = \cdots = (-\frac{1}{5}, -\frac{2\sqrt{6}}{5})$ and $P(4) = P(6) = P(8) = \cdots = (\frac{1}{5}, \frac{2\sqrt{6}}{5})$ are on the line $G(2)G(3)$. For $k \ge 4$, $G(k)$ can be calculated as the center of mass of a point with weight $k-1$ at $G(k-1)$ and one with weight unity at $P(k)$. Hence, $G(k)$ lies on the line $G(2)G(3)$ if $G(k-1)$ does so. Thus, we know inductively that every $G(k)$ is on the line $G(2)G(3)$.

We next show by induction that for odd $k \ge 2$, $G(k)$ is on the segment $OG(2)$. The claim is trivial for $k = 2$. Suppose that $G(k-2)$ lies on the segment $OG(2)$ for some odd $k(\ge 4)$. Consider that two points are added at $P(k-1)$ and $P(k)$ at once. The center of mass of these two points is obviously at the origin. Then, $G(k)$ is regarded as the center of mass of a point with weight $k-2$ at $G(k-2)$ and one with weight two at the origin. Therefore, $G(k)$ is on the segment $OG(2)$.

We can show similarly that for even $k \ge 3$, $G(k)$ is on the segment $OG(3)$. The proof is thus completed. □

Theorem 1. *The strategy $\overline{\boldsymbol{\theta}}$ is optimal for the basic problem. Its competitive difference is $\frac{1}{5}$.*

3.2 Structure of Optimal Offline Strategies

In the proof of Lemma 1, we have claimed that for $k \ge 2$, the strategy $(0, \frac{2\pi}{k}, \frac{4\pi}{k}, \ldots, \frac{2(k-1)\pi}{k})$ achieves a cost of zero. What should be remarked upon here that this is one of optimal offline strategies. A natural question here would be: *What other strategy achieves $C_{opt}(k) = 0$?* In what follows, we do not distinguish strategies with reflection and/or inversion symmetry or those having the same set of angles.

For $k = 2$, there does not exist such a strategy except for the strategy $(0, \pi)$. For $k = 3$, it is seen that $(0, \frac{2\pi}{3}, \frac{4\pi}{3})$ is a unique optimal offline strategy. For $k = 4$, by a basic manipulation of equations, it is derived that any optimal strategy has the form $(0, \pi, \theta_3, \theta_3 + \pi)$. Geometrically speaking, $C_{\boldsymbol{\theta}}(4) = 0$ if and only if $P_{\boldsymbol{\theta}}(1)P_{\boldsymbol{\theta}}(2)P_{\boldsymbol{\theta}}(3)P_{\boldsymbol{\theta}}(4)$ forms a rectangle.

What if $k = 5$? Apparently, the strategy $(0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5})$, which forms a regular pentagon, is optimal. We also have $(0, \pi, \theta_3, \theta_3 + \frac{2\pi}{3}, \theta_3 + \frac{4\pi}{3})$, for which the points compose a diameter and a regular triangle. Note that since the center of mass of a diameter and that of a regular triangle lie both at the origin, the center of mass of the five points lies at the origin as well. Then, is there any strategy that satisfies $C_{\boldsymbol{\theta}}(5) = 0$ but does not form either a regular pentagon or the combination of a diameter and a regular triangle? The answer is yes. For $\frac{\pi}{5} > \varepsilon > 0$, we can choose $\delta > 0$ such that the strategy $(0, \frac{2\pi}{5} - \delta, \frac{4\pi}{5} + \varepsilon, \frac{6\pi}{5} - \varepsilon, \frac{8\pi}{5} + \delta)$ has a cost of zero.

4 The n-Cyclotomic Problem

For each of the cases of odd n and even n, we provide a simple strategy and analyze its competitive difference. Unfortunately, the both strategies are *not* optimal in general. We mention it after each proof.

4.1 Simple Online Strategy for Odd n

We first give a helper lemma for Theorem 2.

Lemma 4. *For $k = 1, \ldots, n$, it holds that*

$$\sum_{j=1}^{k} \cos \frac{(j-1)(n-1)\pi}{n} = \frac{1}{2\cos\frac{\pi}{2n}} \left(\cos\frac{\pi}{2n} + \sin\frac{(2k-1)(n-1)\pi}{2n} \right),$$

and

$$\sum_{j=1}^{k} \sin \frac{(j-1)(n-1)\pi}{n} = \frac{1}{2\cos\frac{\pi}{2n}} \left(\sin\frac{\pi}{2n} - \sin\frac{(2k(n-1)+1)\pi}{2n} \right).$$

Proof. Let

$$c_k := \sum_{j=1}^{k} \cos \frac{(j-1)(n-1)\pi}{n} - \frac{1}{2\cos\frac{\pi}{2n}} \left(\cos\frac{\pi}{2n} + \sin\frac{(2k-1)(n-1)\pi}{2n} \right),$$

and

$$s_k := \sum_{j=1}^{k} \sin \frac{(j-1)(n-1)\pi}{n} - \frac{1}{2\cos\frac{\pi}{2n}} \left(\sin\frac{\pi}{2n} - \sin\frac{(2k(n-1)+1)\pi}{2n} \right).$$

We immediately have $c_1 = 1 - \frac{1}{2\cos\frac{\pi}{2n}} \left(\cos\frac{\pi}{2n} + \cos\frac{\pi}{2n}\right) = 0$. If $c_k = 0$ holds, then

$$\begin{aligned}
c_{k+1} =& c_k + \cos\frac{k(n-1)\pi}{n} + \frac{1}{2\cos\frac{\pi}{2n}} \cdot \left(\sin\frac{(2k+1)(n-1)\pi}{2n} - \sin\frac{(2k-1)(n-1)\pi}{2n} \right) \\
=& \cos\frac{k(n-1)\pi}{n} + \frac{1}{2\cos\frac{\pi}{2n}} \cdot 2\cos\frac{k(n-1)\pi}{n} \sin\left(\frac{\pi}{2n} - \frac{\pi}{2}\right) \\
=& \cos\frac{k(n-1)\pi}{n} - \frac{1}{2\cos\frac{\pi}{2n}} \cdot 2\cos\frac{k(n-1)\pi}{n} \cos\frac{\pi}{2n} \\
=& 0.
\end{aligned}$$

By induction, we conclude $c_k = 0$ for $k = 1, \ldots, n$. Similarly, we have $s_1 = 0 - \frac{1}{2\cos\frac{\pi}{2n}} \left(\sin\frac{\pi}{2n} - \sin\frac{\pi}{2n}\right) = 0$. If $s_k = 0$ holds, then

$$\begin{aligned}
s_{k+1} =& s_k + \sin\frac{k(n-1)\pi}{n} - \frac{1}{2\cos\frac{\pi}{2n}} \cdot \left(-\sin\frac{(2(k+1)(n-1)+1)\pi}{2n} + \sin\frac{(2k(n-1)+1\pi}{2n} \right) \\
=& \sin\frac{k(n-1)\pi}{n} - \frac{1}{2\cos\frac{\pi}{2n}} \cdot 2\cos\left(\frac{k(n-1)\pi}{n} + \frac{\pi}{2}\right) \sin\left(\frac{\pi}{2n} - \frac{\pi}{2}\right) \\
=& \sin\frac{k(n-1)\pi}{n} - \frac{1}{2\cos\frac{\pi}{2n}} \cdot 2\sin\frac{k(n-1)\pi}{n} \cos\frac{\pi}{2n} \\
=& 0.
\end{aligned}$$

Therefore, $s_k = 0$ holds for $k = 1, \ldots, n$. □

Theorem 2. *For the n-cyclotomic problem with odd $n (\geq 3)$, the strategy $\boldsymbol{\theta}$ defined as*

$$\theta_j = \frac{(j-1)(n-1)\pi}{n}, \quad 1 \leq j \leq n$$

achieves a competitive difference of zero for $n = 3$ and $n = 5$, and a competitive difference of $\frac{1}{3\cos\frac{\pi}{14}} (< 0.35)$ for $n \geq 7$.

Proof. For $n = 3$, our strategy is $\boldsymbol{\theta} = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$. By rotational symmetry, there is no choice of strategy. One can easily see that our strategy achieves a competitive difference of zero.

For $n = 5$, we have $\boldsymbol{\theta} = (0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5})$. Observe each time when the k-th item has arrived $(1 \leq k \leq 5)$. One can confirm that there is no better placement than that of our strategy, even if the cardinality is known in advance. Thus, our strategy achieves a competitive difference of zero for $n = 5$.

In the rest of the proof we discuss $n \geq 7$. We calculate the coordinate of the center of mass using Lemma 4. We have

$$\begin{aligned} C_{\boldsymbol{\theta}}(k) =& \frac{1}{2k \cos \frac{\pi}{2n}} \cdot \left(\left(\cos \frac{\pi}{2n} + \sin \frac{(2k-1)(n-1)\pi}{2n} \right)^2 + \left(\sin \frac{\pi}{2n} - \sin \frac{(2k(n-1)+1)\pi}{2n} \right)^2 \right)^{\frac{1}{2}} \\ =& \frac{\left| \sin \frac{k(n-1)\pi}{2n} \right|}{k \cos \frac{\pi}{2n}}. \end{aligned}$$

We investigate the value of $C_{\boldsymbol{\theta}}(k) - C_{opt}(k)$ for all k. For $k = 1$, we have $C_{\boldsymbol{\theta}}(1) = 1$ and obviously $C_{opt}(1) = 1$. Therefore the difference is zero. For $k = 2$, we immediately have $C_{\boldsymbol{\theta}}(2) = \sin \frac{\pi}{2n}$. By a simple calculation, it turns out that to place two points at $(1, 0)$ and $(\cos \frac{(n-1)\pi}{n}, \sin \frac{(n-1)\pi}{n})$ is optimal. Thus, $C_{opt}(2) = \sin \frac{\pi}{2n}$. Hence, the difference is again zero. For $k \geq 3$, by applying $C_{opt}(k) \geq 0$, we have

$$C_{\boldsymbol{\theta}}(k) - C_{opt}(k) \leq C_{\boldsymbol{\theta}}(k) = \frac{\left| \sin \frac{k(n-1)\pi}{2n} \right|}{k \cos \frac{\pi}{2n}}.$$

We derive

$$\frac{\left| \sin \frac{k(n-1)\pi}{2n} \right|}{k \cos \frac{\pi}{2n}} \leq \frac{1}{k \cos \frac{\pi}{2n}} \leq \frac{1}{3 \cos \frac{\pi}{2n}} \leq \frac{1}{3 \cos \frac{\pi}{14}},$$

since $\sin x \leq 1$ for all x, $\cos \frac{\pi}{2n}$ decreases monotonically with n, and $n \geq 7$. The competitive difference of the strategy $\boldsymbol{\theta}$ is thus upper-bounded by $\frac{1}{3 \cos \frac{\pi}{14}} (< 0.35)$ for $n \geq 7$. □

See Figure 2 for the behavior. We here remark that our strategy is not optimal: For $n = 7$, the strategy $(0, \frac{6\pi}{7}, \frac{10\pi}{7}, \frac{4\pi}{7}, \frac{12\pi}{7}, \frac{2\pi}{7}, \frac{8\pi}{7})$ has a competitive difference of zero, while the competitive difference of our strategy $(0, \frac{6\pi}{7}, \frac{12\pi}{7}, \frac{4\pi}{7}, \frac{10\pi}{7}, \frac{2\pi}{7}, \frac{8\pi}{7})$ is approximately 0.08. For $n = 9$, the strategy $(0, \frac{8\pi}{9}, \frac{14\pi}{9}, \frac{4\pi}{9}, \frac{12\pi}{9}, \frac{2\pi}{9}, \frac{6\pi}{9}, \frac{16\pi}{9}, \frac{10\pi}{9})$ has a better competitive difference than that of our strategy $(0, \frac{8\pi}{9}, \frac{16\pi}{9}, \frac{6\pi}{9}, \frac{14\pi}{9}, \frac{4\pi}{9}, \frac{12\pi}{9}, \frac{2\pi}{9}, \frac{10\pi}{9})$. That is to say, our strategy is not optimal for these cases.

In addition, our strategy is not optimal for large n; roughly speaking, a strategy more like that presented in Theorem 1 performs better. More specifically, one can have a better strategy by rounding each position specified in the strategy in Theorem 1 into some nearby position that is feasible for the n-cyclotomic problem, in such a way that each position does not occur more than once. Although the rounded positions for later points may be far from those in the original strategy, this does not matter. Recall that the positions for later points do not affect the competitiveness, as we discussed in Section 3.1.

4.2 Simple Online Strategy for Even n

We have the next theorem for even n.

Theorem 3. *For the n-cyclotomic problem with even $n(\geq 2)$, the strategy $\boldsymbol{\theta}$ defined as*

$$\theta_j = \begin{cases} \frac{(j-1)\pi}{n}, & j \text{ is odd;} \\ \frac{(j-2)\pi}{n} + \pi, & j \text{ is even} \end{cases}$$

achieves a competitive difference of zero for $n = 4$, and a competitive difference of $\frac{1}{3}$ for $n \geq 6$.

Proof. For $n = 4$, the strategy obviously has a competitive difference of zero; there is no choice of strategy.

For $n \geq 6$ we first derive $C_{\boldsymbol{\theta}}(k)$ in a closed form. It is observed that every two angles in $\boldsymbol{\theta}$ place two points so that they form a diameter. Therefore, for even k, the center of mass lies at the origin and $C_{\boldsymbol{\theta}}(k) = 0$. Apparently, $C_{\boldsymbol{\theta}}(1) = 1$. What remains is odd $k \geq 3$. We have already known $C_{\boldsymbol{\theta}}(k-1) = 0$ for such k. The center of mass after placing the k-th point can be considered as the center of mass of the following two weighted points: a point with weight of $k-1$ at the origin and one with unit weight at $P_{\boldsymbol{\theta}}(k)$ on the unit circle. Hence, the center of mass of the k points divides the line segment $OP_{\boldsymbol{\theta}}(k)$ in the ratio $1 : k-1$. Noting $\overline{OP_{\boldsymbol{\theta}}(k)} = 1$, we have $C_{\boldsymbol{\theta}}(k) = \frac{1}{1+(k-1)} = \frac{1}{k}$ for odd k.

We next check the value of $C_{\boldsymbol{\theta}}(k) - C_{opt}(k)$ for all k. For $k = 1$, we have $C_{opt}(1) = 1$ and thus the difference is zero. For odd $k \geq 3$, we obtain

$$C_{\boldsymbol{\theta}}(k) - C_{opt}(k) \leq C_{\boldsymbol{\theta}}(k) = \frac{1}{k} \leq \frac{1}{3},$$

since $C_{opt}(k) \geq 0$ holds. For even $k \geq 2$, we have

$$C_{\boldsymbol{\theta}}(k) - C_{opt}(k) \leq C_{\boldsymbol{\theta}}(k) = 0.$$

We thus conclude that the competitive difference of the strategy $\boldsymbol{\theta}$ is at most $\frac{1}{3}$. □

See Figure 3 for the behavior. We add without proof that not only for $n = 4$ but also for $n = 6$, 8, and 10, our strategy is an optimal strategy. The competitive difference is $\frac{1}{3}$ for $n = 6$, $\frac{\sqrt{2}-1}{3} (\approx 0.20)$ for $n = 8$, and $\frac{\sqrt{5}-1}{6} (\approx 0.21)$ for $n = 10$. For large n, however, it turns out that our strategy is not optimal by the same reason as for large odd n.

4.3 Conditions for $C_{opt}(k) = 0$

Unlike in the basic problem, in the n-cyclotomic problem $C_{opt}(k) = 0$ is not always true for $k \geq 2$. Apart from online optimization, there arises an interesting question: *Which pair (n, k) admits $C_{opt}(k) = 0$?* In this subsection we give a partial answer. We start from easy cases.

Lemma 5. *For any n, $C_{opt}(1) = 0$, $C_{opt}(n-1) = \frac{1}{n-1}$, and $C_{opt}(n) = 0$.*

Proof. $C_{opt}(1) = 0$ and $C_{opt}(n) = 0$ are trivial. We now see why $C_{opt}(n-1) = \frac{1}{n-1}$ holds. Suppose that $n-1$ points have been placed optimally (though there is no choice) and their center of mass $G(n-1)$ lies somewhere. Next, add a point at $P(n)$, which is the unique destination without a point yet. Then, needless to say, the new center of mass comes to the origin O. By considering that the mass of the $n-1$ points concentrates at $G(n-1)$, the new center can also be thought of as the position that divides the line segment $G(n-1)P(n)$ in the ratio $1 : n-1$. Since $\overline{OP(n)} = 1$, we obtain $C_{opt}(n-1) = \overline{OG(n-1)} = \frac{1}{n-1}$. □

The next theorem gives a sufficient condition when n belongs to a class of composite numbers.

Table 1: Values of $C_{opt}(k)$ for the n-cyclotomic problem ($2 \leq n \leq 12$).

	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$n=2$	1	0										
3	1	0.50	0									
4	1	0	0.33	0								
5	1	0.31	0.21	0.25	0							
6	1	0	0	0	0.2	0						
7	1	0.22	0.19	0.14	0.09	0.17	0					
8	1	0	0.14	0	0.08	0	0.14	0				
9	1	0.17	0	0.09	0.07	0	0.05	0.13	0			
10	1	0	0.13	0	0	0	0.05	0	0.11	0		
11	1	0.14	0.10	0.06	0.02	0.01	0.03	0.04	0.03	0.10	0	
12	1	0	0	0	0	0	0	0	0	0	0.09	0

Theorem 4. *For n even and divisible by some odd number $p \geq 3$, $C_{opt}(k) = 0$ holds if k is even or $p \leq k \leq n - p$.*

Proof. Observe that if some set of placed points forms a diameter of the unit circle or a regular polygon, then the center of mass of the points lies at the origin. The idea of our proof is thus to give a strategy that places points in such a way that they can be decomposed into such sets. If k is even, we can choose $\frac{k}{2}$ pairs of positions that form $\frac{k}{2}$ distinct diameters and the proof is done.

In what follows, assume that k is odd and satisfies $p \leq k \leq n-p$. We present a strategy for such k. For ease of presentation, only the angles appearing in the strategy are described below. Although we give a series of angles with length $n-p$ in total, the strategy is constructed so that to apply only the first k angles always leads to a cost of zero. Let $m = \frac{n}{2p}$. Intuitively, our strategy first makes a regular p-gon followed by $(m-1)p$ distinct diameters. Formally, our strategy is to: (i) Place points at

$$0, \frac{2 \cdot 2m\pi}{n}, \frac{2 \cdot 4m\pi}{n}, \ldots, \frac{2 \cdot 2(p-1)m\pi}{n}.$$

(ii) Then place points, repeatedly for $j = 1, 2, \ldots, p$, at

$$\frac{2((j-1)m+1)\pi}{n}, \frac{2((j-1)m+1)\pi}{n} + \pi,$$
$$\frac{2((j-1)m+2)\pi}{n}, \frac{2((j-1)m+2)\pi}{n} + \pi,$$
$$\ldots, \frac{2((j-1)m+m-1)\pi}{n}, \frac{2((j-1)m+m-1)\pi}{n} + \pi.$$

It is easy to see that the placement of (i) forms a regular p-gon; the difference of the angles is all $\frac{4m\pi}{n} = \frac{2\pi}{p}$. Now $k \geq p$ is assumed, the regular p-gon is always completed.

One can see that in (ii), every pair of angles taken from the head forms a diameter. Since k and p are odd, it does not occur that at the end a diameter is left uncompleted.

Besides, it is seen that in (ii), each iteration with respect to j consists of $m-1$ distinct diameters. What remains is to claim that any angle in (ii) does not coincide with the angles in (i). Note that $\frac{2((j-1)m+l)\pi}{n} + \pi = \frac{2((j-1)m+l+pm)\pi}{n}$. For $l = 1, 2, \cdots, m-1$, both $(j-1)m+l$ and $(j-1)m+l+pm$ are indivisible by m, which implies that none of the angles in (ii) has appeared in (i). □

See Figure 4 for the behavior of the strategy for $n = 40$, $p = 5$, and $k = 15$. Together with Lemma 5, we have a corollary.

Corollary 1. *For n divisible by six, $C_{opt}(k) = 0$ holds if and only if $2 \le k \le n-2$ or $k = n$.*

For the case that n is a prime number, we show that $C_{opt}(k)$ cannot be zero unless $k = n$ through algebraic arguments. We here regard $\mathbb{R}^2$ as the complex plane. Let ζ be $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then, the n destinations are expressed as $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$. The following is a basic fact concerning them.

Fact 1. *(For example, [vdW91]) Any element of $\mathbb{Q}(\zeta)$, an adjunction of ζ to the field of rational numbers, is uniquely expressed in a linear combination of $1, \zeta^1, \zeta^2, \ldots, \zeta^{n-2}$ with $a_j \in \mathbb{Q}$,*

$$a_1 + a_2\zeta + a_3\zeta^2 + \cdots + a_{n-1}\zeta^{n-2}.$$

It is known that when n is a prime number, ζ does not satisfy an equation of degree lower than $n-2$. Therefore, any ζ^k in a linear combination cannot be replaced by a linear combination of others. The fact is explained in terms of linear algebra; $1, \zeta, \zeta^2, \ldots, \zeta^{n-2}$ are basic vectors in that vector space.

Theorem 5. *For n prime, $C_{opt}(k) = 0$ holds if and only if $k = n$.*

Proof. Observe that the linear combination in Fact 1 for $a_j \in \{0, \frac{1}{k}\}$ with $\sum_{j=1}^{n-1} a_j = 1$ represents the center of mass of k points with $1 \le k \le n-1$, placed at k distinct destinations. Apply Fact 1 with the element to be expressed being 0. Then, the lemma states that we have to set $a_1, a_2, \ldots, a_{n-1}$ to all zero. Back in the context of $\mathbb{R}^2$, this fact implies that wherever we place fewer than n points, the center of mass does not come to the origin. □

By a brute-force search on a computer, the numerical value of $C_{opt}(k)$ for the n-cyclotomic problem is calculated as shown in Table 1.

5 Concluding Remarks

Many questions are left open: What if arbitrary weights are allowed? Another measure of balance? How about in $\mathbb{R}^3$ or an arbitrary metric space? (As introduced, there are numerous studies on satellite constellation design such as [dWSS08, Uly09].) What if the destination of points is arbitrarily restricted? For the n-cyclotomic problem, can a more sophisticated strategy be designed? Can the problem in Section 4.3 be solved for general composite numbers?

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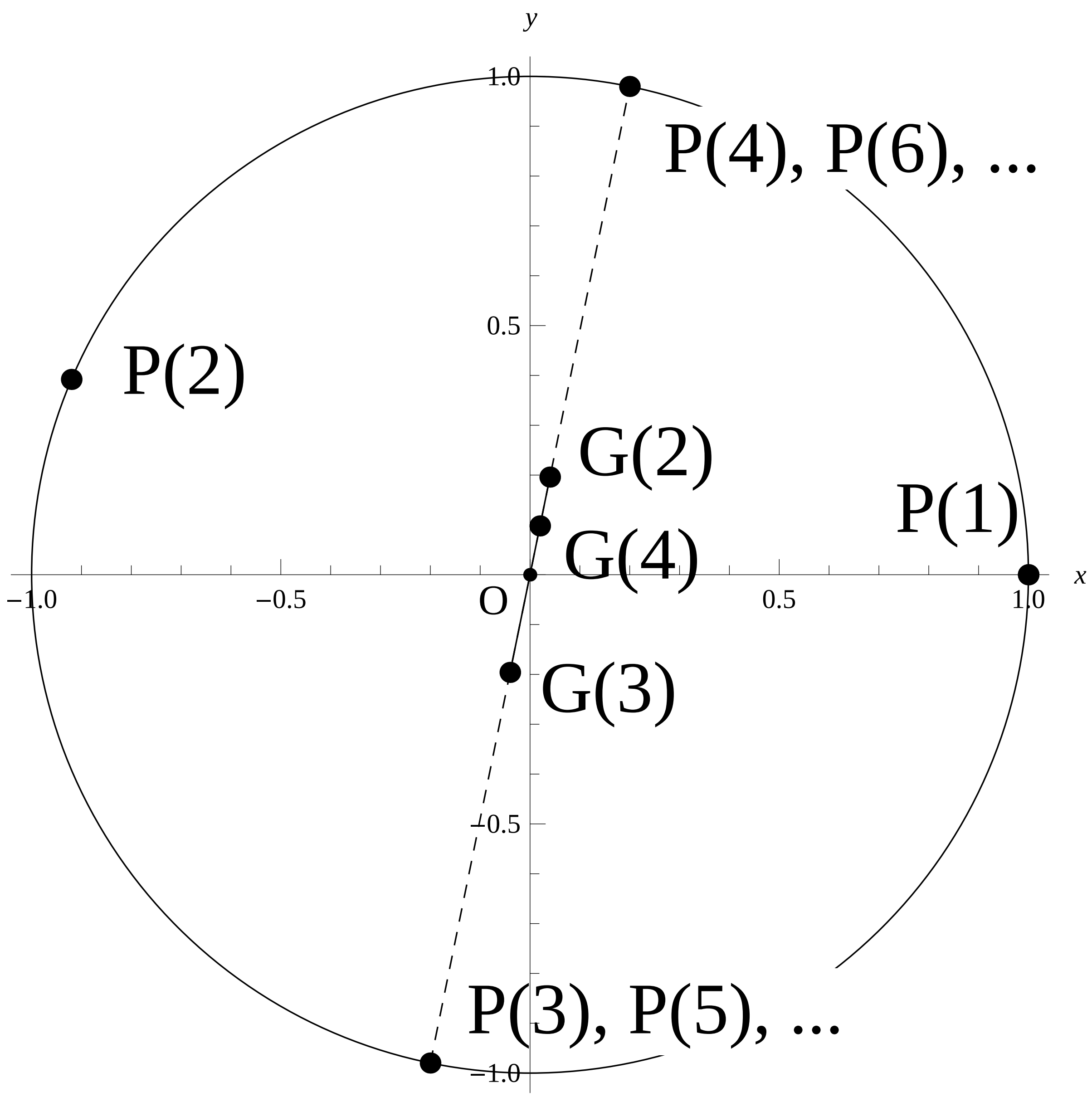

Figure 1: Placement of points $P(1), P(2), \ldots$ according to the optimal online strategy $\overline{\boldsymbol{\theta}}$ in Theorem 1 for the basic problem. $G(i)$ is the center of mass when i points have been placed so far.

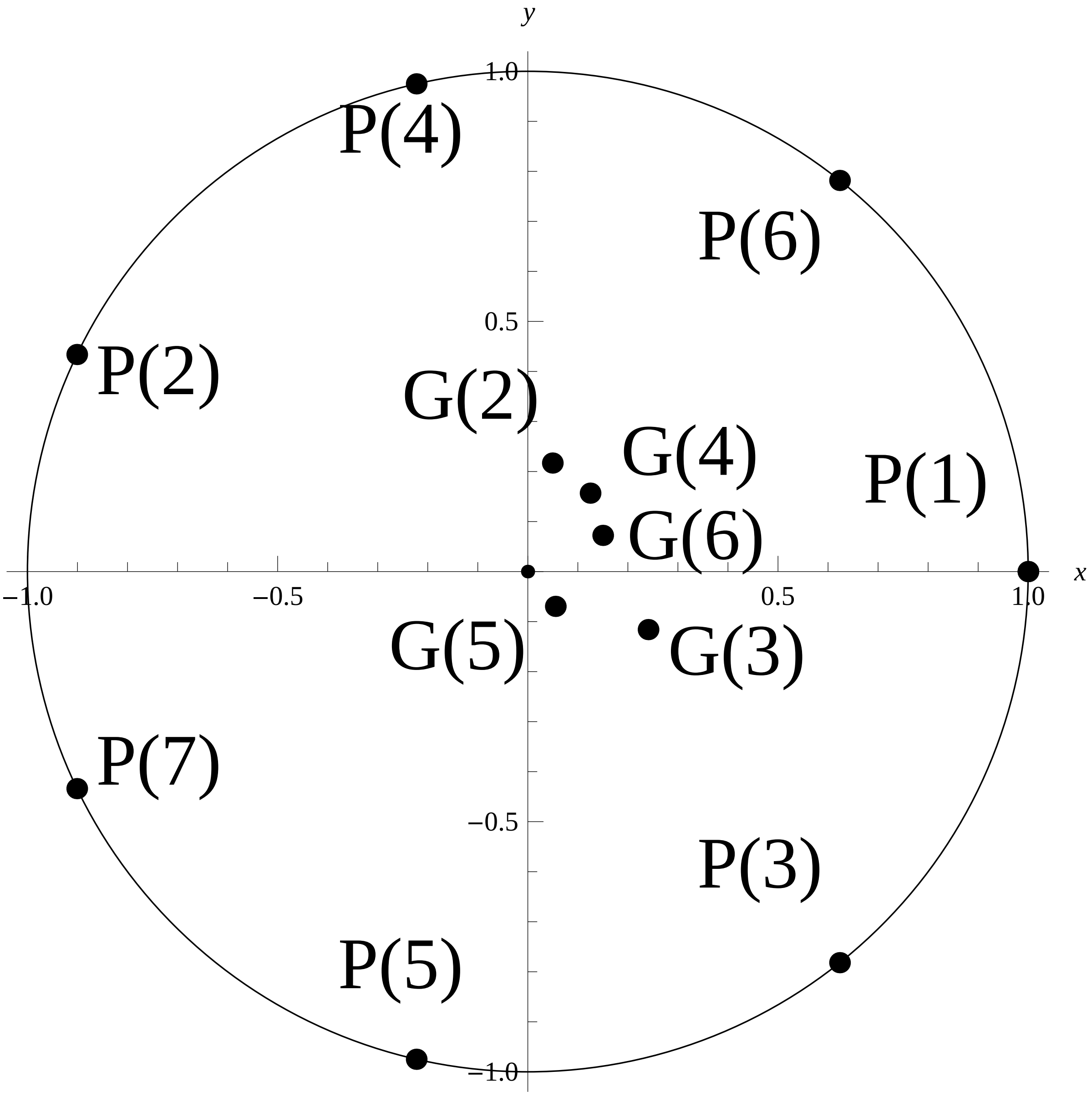

Figure 2: Behavior of the online strategy in Theorem 2 for the 7-cyclotomic problem.

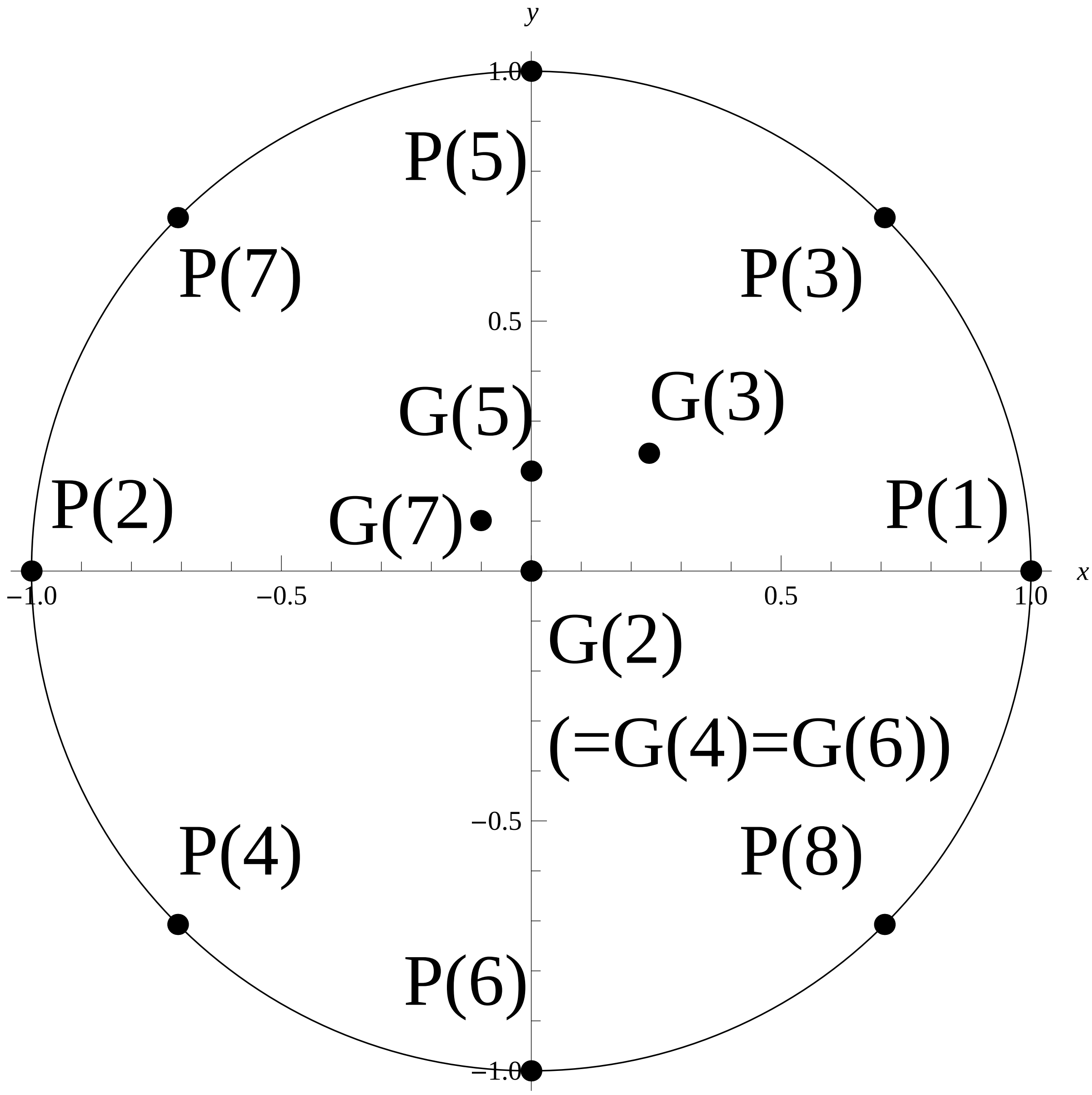

Figure 3: Behavior of the online strategy in Theorem 3 for the 8-cyclotomic problem.

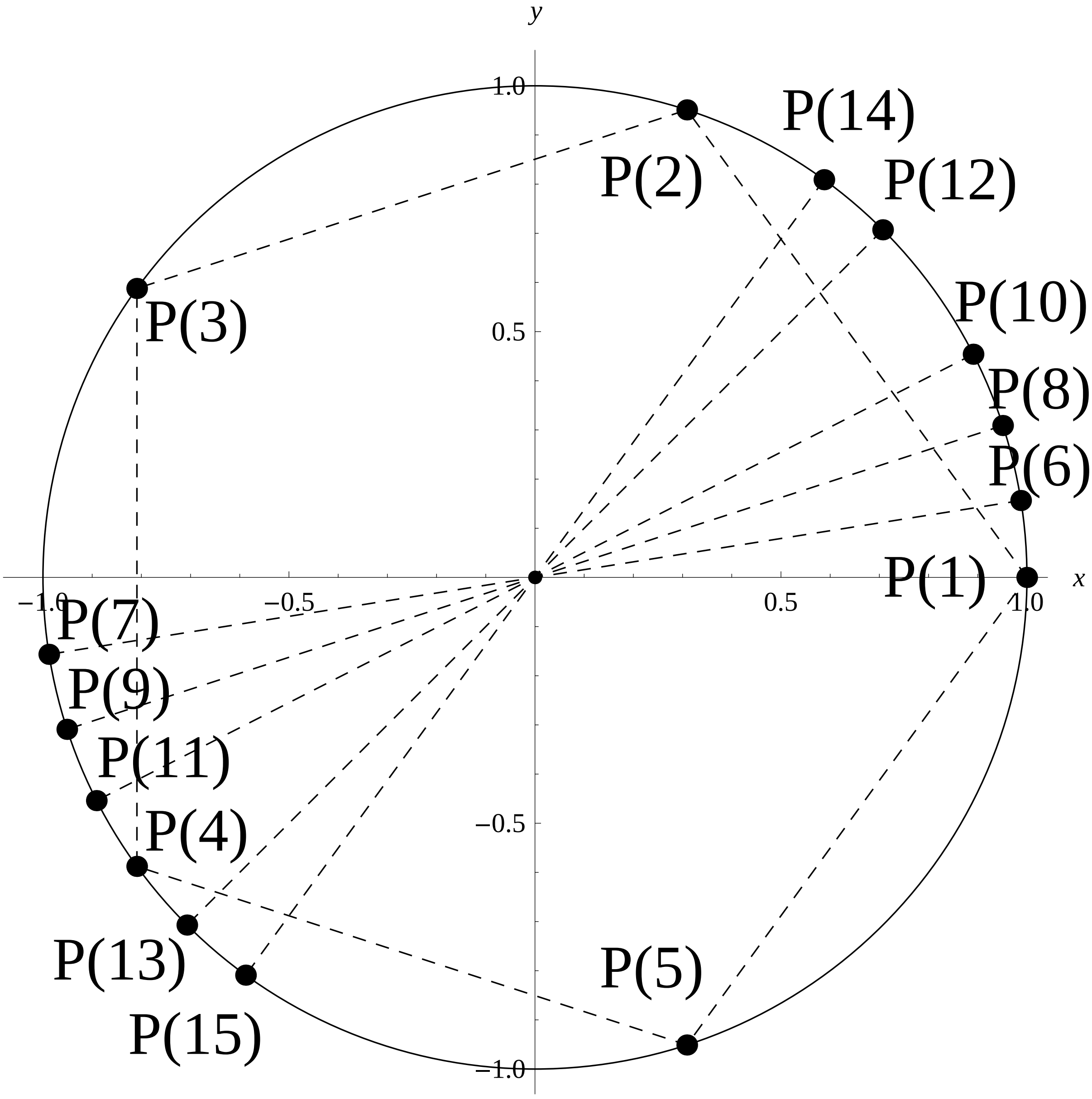

Figure 4: Behavior of the offline strategy in Theorem 4 for the 40-cyclotomic problem, applying $p = 5$. This figure depicts the placement of 15 points such that their center of mass lies at the origin.