On the Best Possible Competitive Ratio for Multislope Ski Rental[1]

Hiroshi Fujiwara, Takuma Kitano, and Toshihiro Fujito
Toyohashi University of Technology

Abstract

The multislope ski-rental problem [LPSR08a] is an extension of the classical ski-rental problem [KMRS88], where the player has several *lease* options in addition to the pure *rent* and *buy* options. In this problem an *instance*, which is the setting of the options, significantly affects the player's performance. There is an algorithm that for a given instance, computes the best possible strategy [AIS08]. However, the output is given in numerical values and the relational nature between the instance and the best possible performance has not been known. In this paper we prove that even for the easiest instance, a competitive ratio smaller than $e/(e-1) \approx 1.58$ cannot be achieved. More precisely, according to the number of options, tight bounds are obtained in a closed form. Furthermore, we establish a matching upper and lower bound of the competitive ratio each for the 3-slope and 4-slope problems.

1 Introduction

The *multislope ski-rental problem* [LPSR08a] is an extension of the classical ski-rental problem [KMRS88], where the player has several *lease* options in addition to the pure *rent* and *buy* options. For example, a store may offer the following options for ski sets: Rent for \$50 per day, or buy for \$500, or lease for \$30 per day with initial fee of \$100, or another lease for \$15 per day with initial fee of \$250. Every time going skiing, the skier chooses one of these options, or keeps on the current choice.

We refer to a set of such options as an *instance*, denoted by $(\boldsymbol{r}, \boldsymbol{b})$ (the detailed definition shall appear in Section 2.) A *strategy* of the skier specifies when and to which option to switch. According to the standard definition, we say that a strategy achieves a *competitive ratio* of c if the skier along the strategy is charged at most c times the optimal offline cost, i.e., one with the number of times of skiing known in advance. For a given instance $(\boldsymbol{r}, \boldsymbol{b})$, we define the *best possible competitive ratio* $\tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ as the minimum value of the competitive ratios for all the possible strategies for $(\boldsymbol{r}, \boldsymbol{b})$.

The numerical value of $\tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ can be calculated by the algorithm of Augustine et al. [AIS08]. However, almost nothing has been known of the dependencies between $(\boldsymbol{r}, \boldsymbol{b})$ and $\tilde{c}(\boldsymbol{r}, \boldsymbol{b})$. In this paper we analyze $\inf \tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ and $\sup \tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ over reasonable instances, revealing the easiest and the hardest instance. Notice here that the supremum coincides with the matching upper and lower bound of the competitive ratio in the standard sense.

The analysis of the infimum is motivated by the following argument: This problem can be regarded as Dynamic Power Management [ISG03] on a system that has multiple *energy-saving states*, for example, a Windows computer with *Stand By* state, *Hibernate* state, and so on. The objective is to minimize the energy consumption while there is no user response for an uncertain duration. A pair of a strategy and an instance that achieve the infimum make the best specification of energy-saving states and the best state-transition schedule.

Our Contribution. (I) For fixed $k \geq 2$, we prove that $\inf\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k)\} = (k+1)^k/((k+1)^k - k^k)$, where $I(k)$ denotes the set of $(k+1)$-*slope* instances having $k+1$ options. (For example, the above instance is in $I(3)$.) The infimum value monotonically decreases as k

[1]This work was partially supported by KAKENHI (23700014 and 23500014).

grows. For example, 1.80 for $k = 2$, 1.73 for $k = 3$, and 1.70 for $k = 4$. We have a corollary: $\inf\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k), k \geq 2\} = e/(e-1) \approx 1.58$. The results are interpreted into the context of Dynamic Power Management as follows: The more energy-saving states are available, the better energy saving performance can be achieved. Nevertheless, there is a limit of improvement.

(II) We show $\sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(2)\} \approx 2.47$ and $\sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(3)\} \approx 2.75$. Please recall that the supremum is the matching upper and lower bound of the competitive ratio in the ordinary sense. The results establish a matching bound each for the 3-slope and 4-slope problems. Our analysis illustrates a different technique of seeking a matching bound from that in the conventional research, where an upper bound and a lower bound are separately considered. The results in (I) and (II) are graphically summarized in Figure 1.

(III) We consider two subclasses of instances $I_A(k)$ and $I_I(k)$ which consist of *additive* instances and *investment* instances, respectively. (The definitions shall be given in Section 2.) We show $\sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I_A(k)\} = 2$ and $\inf\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I_I(k)\} = 2$ for any $k \geq 2$.

Related Work. We first mention studies on the deterministic model. The classical (i.e., 2-slope) ski-rental problem [KMRS88] was first introduced in the context of snoopy caching. The problem admits an optimal 2-competitive strategy. Various practical applications can be found in [Kar96], such as context switching and virtual circuit management. In [ISG03] the multislope ski-rental problem was discussed as Dynamic Power Management. Augustine et al. [AIS08] developed an algorithm that for a given instance, outputs the best possible strategy and its competitive ratio. Bejerano et al. [BCN00] provided a 4-competitive strategy for an arbitrary instance. Although their strategy was originally for an investment instance, the competitiveness straightforwardly applies to an arbitrary instance as well. Irani et al. [ISG03] gave a 2-competitive strategy for an additive instance. For investment instances, Damaschke [Dam03] gave a lower bound of $(5 + \sqrt{5})/2 (\approx 3.62)$. The Bahncard problem [Fle01] is another extension of the 2-slope ski-rental problem.

Karlin et al. [KMMO94] provided an optimal randomized strategy for the 2-slope ski-rental problem with a competitive ratio of $e/(e-1)$. Karlin et al. [KKR01] applied this to several problems including TCP Acknowledgment. El-Yaniv et al. [EYKL99] studied the problem in a market with interest rates. Lotker et al. [LPSR08b] analyzed a 2-slope problem with buy and lease options. For the multislope ski rental, in their other paper [LPSR08a] they presented an e-competitive randomized strategy for an arbitrary instance. Besides, they gave an $e/(e-1)$-competitive randomized strategy for an additive instance, and also an algorithm that computes the best possible randomized strategy for a given additive instance.

2 Problem Statement and Preliminaries

A $(k+1)$-*slope ski-rental instance* consists of a pair of two vectors $(\boldsymbol{r}, \boldsymbol{b})$ that specifies $k+1$ *states* which we have called options in Section 1. Each state is associated with a per-time fee r_i. Hereafter we deal with the number of times of skiing as a nonnegative real number, so the per-time fee is regarded as a fee charged per unit time. A transition from state i to j can be done by paying $b_{i,j}$. There are two special states: State 0 with $b_{0,0} = 0$ and state k with $r_k = 0$, which correspond to *rent* and *buy*, respectively. States $1, \ldots, k-1$ represent *lease* options of paying both a per-time and an initial fees. Without loss of generality we assume that the player starts from state 0 at time 0; he/she may transition to another state immediately. The problem with $k = 1$, i.e., with no lease states, is equivalent to the classical ski-rental problem.

The example of options at the beginning of Section 1 is now described as $(r_0, r_1, r_2, r_3) = (1, 0.6, 0.3, 0)$ and $(b_{0,1}, b_{0,2}, b_{0,3}, b_{1,2}, b_{1,3}, b_{2,3}) = (0.2, 0.5, 1, 0.3,\ 0.8, 0.5)$, after normalizing all the values so that (i) to buy a ski set costs 1 and (ii) to keep renting it by time 1 costs 1. Since our interest is not in the absolute cost but in the ratio of costs, these normalizations do not lose

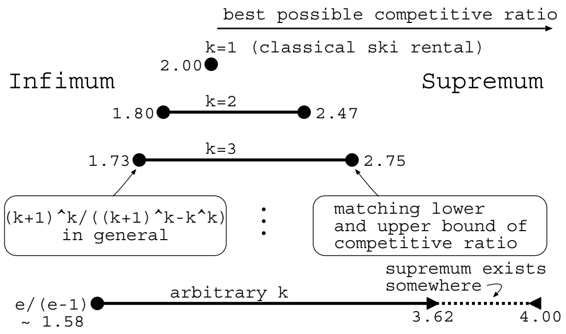

Figure 1: Illustration of the range of the best possible competitive ratio for $(k+1)$-slope ski-rental problem. The infimum/supremum is achieved by the easiest/hardest instance, respectively.

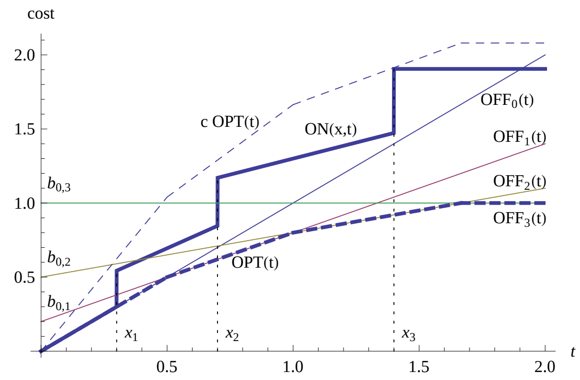

Figure 2: Cost functions for a 4-slope ski-rental instance. $ON(\boldsymbol{x}, t)$ jumps at $t = x_1$, x_2, and x_3 due to state transitions. Elsewhere it increases linearly with slope r_i. $OPT(t)$ transitions to an optimal state at the beginning.

generality. In this example each transition cost between states is set as the difference of their initial costs. Figure 2 illustrates cost functions (introduced soon) based on this example.

We define the set of instances $I(k)$ to be the set of $(\boldsymbol{r}, \boldsymbol{b})$ such that:

$$1 = r_0 > r_1 > \cdots > r_k = 0, \quad 0 = b_{0,0} \le b_{0,1} \le \cdots \le b_{0,k} = 1,$$

$$b_{l,j} - b_{l,i} \le b_{i,j} \le b_{l,j} \text{ for } 0 \le l < i < j \le k, \tag{1}$$

$$b_{0,i+1}\left(-r_{i-1} + r_i\right) + b_{0,i}\left(r_{i-1} - r_{i+1}\right) + b_{0,i-1}\left(-r_i + r_{i+1}\right) \le 0 \text{ for } 1 \le i \le k-1. \tag{2}$$

$\{r_i\}$ is decreasing and $\{b_{0,i}\}$ is non-decreasing. The left inequality in (1) is a constraint that a transition cost cannot be saved by going through another state. Unless the right inequality in (1) holds, a transition from state i to j is more expensive than a transition from a state with a smaller index than i. Refer to [AIS08] for the original motivation of these restrictions in Dynamic Power Management. We may thus assume that the player never transitions back to a state with a smaller index and so do not define $b_{j,i}$ $(0 \le i < j \le k)$. The reason for (2) shall be clarified after the introduction of cost functions.

An *additive* instance in $I_A(k) \subset I(k)$ is such that $b_{i,j} = b_{i,l} + b_{l,j}$ holds for all $0 \le i < l < j \le k$: For a transition the player pays just the difference of the "potential" of state. One can confirm that the above example is in $I_A(3)$. An *investment* instance in $I_I(k) \subset I(k)$ is such that $b_{i,j} = b_{0,j}$ is satisfied for all $0 \le i < j \le k$: Each transition cost is independent of its origin.

A deterministic *strategy* of the player is a vector $\boldsymbol{x}$ with $k+1$ entries. Each entry x_i stands for the time when the player transitions to state i. Since the player loses on a backward transition, the sequence of the entries is assumed to be non-decreasing. Also, we can fix $x_0 = 0$ even if the player goes to state $i > 0$ at time 0. The whole set of strategies is thus written as $S = \{\boldsymbol{x} \mid 0 = x_0 \le x_1 \le \cdots \le x_k\}$. The player may transition from state i directly to j by skipping the states between. Then, we set $x_{i+1} = \cdots = x_{j-1} = x_j$ and define a relation of $i \prec j$. The online player with a strategy $\boldsymbol{x}$ will have paid a cost of

$$ON(\boldsymbol{x}, t) := r_i\left(t - x_i\right) + \sum_{l=0}^{i-1} r_l\left(x_{l+1} - x_l\right) + \sum_{0 \le l \prec m \le i} b_{l,m}$$

by time t ($x_i < t < x_{i+1}$). For t being exactly a transition time x_i, we define the function as the cost immediately after the transition: $ON(\boldsymbol{x}, x_i) := \sum_{l=0}^{i-1} r_l (x_{l+1} - x_l) + \sum_{0 \le l \prec m \le i} b_{l,m}$.

The *optimal offline player* plays optimally with t known. It is observed that the optimal strategy is to transition to a state at time 0 and then keep staying there. The cost is

$$OPT(t) := \min_{0 \le j \le k} OFF_j(t),$$

where $OFF_j(t) := r_j t + b_{0,j}$ represents the cost of staying at state j from time 0 to t. In other words, $OPT(t)$ is the lower envelope of $\{OFF_i(t)\}$.

We now explain the reason of the condition (2) of $I(k)$. This means that every line $OFF_i(t)$ constitutes some part of $OPT(t)$, which is equivalent to the fact that there is no state that the optimal offline player never uses.

For an instance $(\boldsymbol{r}, \boldsymbol{b})$, a strategy $\boldsymbol{x}$ is said to be *c-competitive* if $ON(\boldsymbol{x}, t) - c \cdot OPT(t) \le 0$ for all $t \ge 0$. c is called the *competitive ratio.* Competitiveness can be explained visually in Figure 2. If $ON(\boldsymbol{x}, t)$ is drawn below $c \cdot OPT(t)$, then strategy $\boldsymbol{x}$ is c-competitive. We define the *best possible competitive ratio*

$$\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) := \inf\{c \mid (\forall t \ge 0)\ ON(\boldsymbol{x}, t) - c \cdot OPT(t) \le 0, \boldsymbol{x} \in S\}$$

for $(\boldsymbol{r}, \boldsymbol{b})$. The condition can be weaken by the following two arguments. The first is a lemma that allows us to concentrate on strategies satisfying $ON(\boldsymbol{x}, t) = c \cdot OPT(t)$ at every transition time. Such a strategy is called *eager* in [AIS08].

Lemma 1 ([AIS08])**.** *Suppose that a strategy $\boldsymbol{x}$ is c-competitive. Then, there is a c-competitive strategy $\boldsymbol{x'}$ such that $ON(\boldsymbol{x'}, x'_i) = c \cdot OPT(x'_i)$ holds for all $0 \le i \le k$.*

The second argument is on $OPT(t)$. The minimum operation can be eliminated by imposing $k+1$ constraints instead. Hence, we will hereafter employ

$$\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) = \inf\{c \mid (0 \le \forall i \le k, 0 \le \forall j \le k)\ ON(\boldsymbol{x}, x_i) - c \cdot OFF_j(x_i) \le 0, \boldsymbol{x} \in S\}.$$

The next theorem enables us to calculate the value of $\tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ within arbitrary precision. However, the analytical relation between $(\boldsymbol{r}, \boldsymbol{b})$ and $\tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ has not been known. In subsequent sections we analyze $\inf \tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ and $\sup \tilde{c}(\boldsymbol{r}, \boldsymbol{b})$, revealing the easiest and the hardest instance.

Theorem 1 ([AIS08])**.** *For any $\varepsilon > 0$, there exists an algorithm that for given $(\boldsymbol{r}, \boldsymbol{b})$, computes a $(\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) + \varepsilon)$-competitive strategy in $O(k^2 \log k \log(1/\varepsilon))$ time.*

3 Infimum of the Best Possible Competitive Ratio

Fixed k. The infimum is written as

$$\begin{aligned}&\inf\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k)\} \\ &\quad = \inf\{c \mid (0 \le \forall i \le k, 0 \le \forall j \le k)\ ON(\boldsymbol{x}, x_i) - c \cdot OFF_j(x_i) \le 0, \boldsymbol{x} \in S, (\boldsymbol{r}, \boldsymbol{b}) \in I(k)\}.\end{aligned}$$

The simple lemma below implies that there is an instance in $I_A(k)$ which achieves the infimum. The proof will appear in the full version.

Lemma 2. *Let $(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{b}, c)$ be such that $\boldsymbol{x} \in S$, $(\boldsymbol{r}, \boldsymbol{b}) \in I(k)$, and $ON(\boldsymbol{x}, x_i) - c \cdot OFF_j(x_i) \le 0$ for all $0 \le i \le k$ and $0 \le j \le k$. Set $\boldsymbol{b'}$ as $b'_{i,j} := b_{0,j} - b_{0,i}$ for $0 < i < j \le k$ and $b'_{0,i} := b_{0,i}$ for $0 \le i \le k$. Then, also for $(\boldsymbol{r}, \boldsymbol{b'}) \in I_A(k)$, $ON(\boldsymbol{x}, x_i) - c \cdot OFF_j(x_i) \le 0$ holds for all $0 \le i \le k$ and $0 \le j \le k$.*

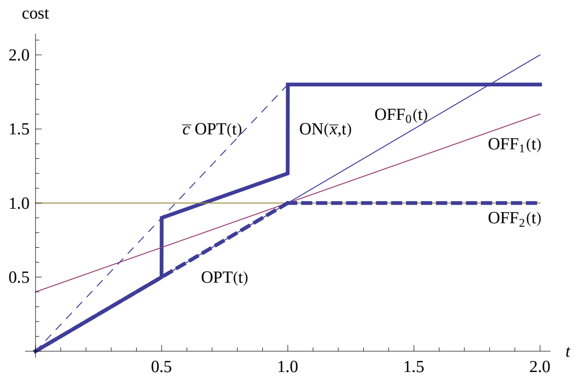

Figure 3: Instance and strategy that achieve the infimum for $k = 2$.

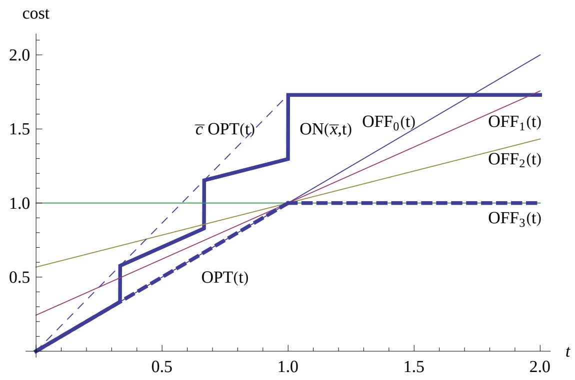

Figure 4: Instance and strategy that achieve the infimum for $k = 3$.

For an additive instance the second term of $ON(\boldsymbol{x}, x_i)$ equals $b_{0,i}$, since the sum of transition costs is independent of through which states the online player has passed. So, let

$$g_{i,j}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{b}, c) := ON(\boldsymbol{x}, x_i) - c \cdot OFF_j(x_i) = \sum_{l=0}^{i-1} r_l (x_{l+1} - x_l) + b_{0,i} - c(r_j x_i + b_{0,j}).$$

We thus formulate a mathematical program with variables $\boldsymbol{x}$, $\boldsymbol{r}$, $\boldsymbol{b}$, and c.

$$(\mathcal{P}) \quad \text{minimize } c$$
$$\text{subject to } g_{i,j}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{b}, c) \leq 0, \qquad \text{for } 0 \leq i \leq k, 0 \leq j \leq k,$$
$$\boldsymbol{x} \in S, (\boldsymbol{r}, \boldsymbol{b}) \in I_A(k).$$

In spite of its nonconvexity, we solve the problem analytically and obtain an explicit solution. The infimum is indeed the minimum.

Theorem 2. *The following* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{r}}, \overline{\boldsymbol{b}}, \overline{c})$ *is a global optimum to problem* $(\mathcal{P})$*:*

$$\overline{x}_i = \frac{i}{k}, \qquad \textit{for } 0 \leq i \leq k, \tag{3}$$

$$\overline{r}_i = \overline{c} + (1 - \overline{c})\left(1 + \frac{1}{k}\right)^i, \qquad \textit{for } 0 \leq i \leq k, \tag{4}$$

$$\overline{b}_{0,i} = 1 - \overline{r}_i, \qquad \textit{for } 0 \leq i \leq k, \tag{5}$$

$$\overline{b}_{i,j} = \overline{b}_{0,j} - \overline{b}_{0,i}, \qquad \textit{for } 0 < i < j \leq k, \tag{6}$$

$$\overline{c} = \frac{(k+1)^k}{(k+1)^k - k^k}. \tag{7}$$

Corollary 1. $\inf\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k)\} = \min\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k)\} = (k+1)^k/((k+1)^k - k^k)$.

Before going to the proof, we give some numerical examples. For $k = 2$, we have $\overline{c} = \frac{9}{5} = 1.80$, $(\overline{x}_0, \overline{x}_1, \overline{x}_2) = (0, \frac{1}{2}, 1)$, $(\overline{r}_0, \overline{r}_1, \overline{r}_2) = (1, \frac{3}{5}, 0)$, and $(\overline{b}_{0,1}, \overline{b}_{0,2}, \overline{b}_{1,2}) = (\frac{2}{5}, 1, \frac{3}{5})$. And for $k = 3$, $\overline{c} = \frac{64}{37} \approx 1.73$, $(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3) = (0, \frac{1}{3}, \frac{2}{3}, 1)$, $(\overline{r}_0, \overline{r}_1, \overline{r}_2, \overline{r}_3) = (1, \frac{28}{37}, \frac{16}{37}, 0)$, and $(\overline{b}_{0,1}, \overline{b}_{0,2}, \overline{b}_{0,3}, \overline{b}_{1,2}, \overline{b}_{1,3}, \overline{b}_{2,3}) = (\frac{9}{37}, \frac{21}{37}, 1, \frac{12}{37}, \frac{28}{37}, \frac{16}{37})$. Figures 3 and 4 show cost functions based on these instances and strategies. Please recall that $OPT(t)$ is the lower envelope of $\{OFF_i(t)\}$.

One can see that in each figure, all other $OFF_i(t)$ than $OFF_0(t)$ and $OFF_k(t)$ degenerates to one point. This fact is interpreted that the choice of the offline player is narrowed down to either of state 0 or k. On the other hand, the resulting strategy keeps transitioning to the next state at equal time intervals while exploiting all the states.

Our original interest was how much the online player can get advantage for the easiest instance. However, the resulting situation seems a bit different: The instance is unfavorable to the offline player, rather than easy to the online player. This would be a limit of worst-case competitive analysis.

Arbitrary k. We have

$$\overline{c} = 1 + \frac{1}{\left(1+\frac{1}{k}\right)^k - 1} \to 1 + \frac{1}{e-1} = \frac{e}{e-1}$$

as $k \to \infty$, which provides the following corollary in a straightforward manner. The corollary is more interesting in Dynamic Power Management: Even if arbitrarily many energy-saving states can be implemented, there is a limit of improvement.

Corollary 2. $\inf\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k), k \geq 2\} = e/(e-1) \approx 1.58$. *That is to say, for any instance, no strategy achieves a competitive ratio* $\leq e/(e-1)$.

Proof of Theorem 2. In this proof we will state some lemmas whose proofs will be given in the full version. First, by regarding $\boldsymbol{r}$ in problem $(\mathcal{P})$ as a parameter, denoted by $\boldsymbol{r}^*$, we obtain a parametric optimization problem $(\mathcal{Q})$.

$$\begin{aligned}
(\mathcal{Q}) \quad & \text{minimize } c \\
& \text{subject to } g_{i,j}(\boldsymbol{x}, \boldsymbol{r}^*, \boldsymbol{b}, c) \leq 0, \qquad \text{for } 0 \leq i \leq k, 0 \leq j \leq k, \\
& \qquad \boldsymbol{x} \in S, (\boldsymbol{r}^*, \boldsymbol{b}) \in I_A(k).
\end{aligned}$$

We guess a solution to problem $(\mathcal{Q})$. Set $b^*_{0,i} := 1 - r^*_i$ for $0 \leq i \leq k$ and then $b^*_{i,j} := b^*_{0,j} - b^*_{0,i}$ for $0 < i < j \leq k$. We solve the equations $g_{i,0}(\boldsymbol{x}, \boldsymbol{r}^*, \boldsymbol{b}^*, c) = 0$ for $0 \leq i \leq k$ and $g_{k,k}(\boldsymbol{x}, \boldsymbol{r}^*, \boldsymbol{b}^*, c) = 0$. By taking the difference, we have for each $1 \leq i \leq k$,

$$g_{i+1,0}(\boldsymbol{x}, \boldsymbol{r}^*, \boldsymbol{b}^*, c) - g_{i,0}(\boldsymbol{x}, \boldsymbol{r}^*, \boldsymbol{b}^*, c) = -\left(x_{i+1} - x_i\right)\left(c - r^*_i\right) + r^*_i - r^*_{i+1} = 0.$$

Summing up $x_{i+1} - x_i$ yields

$$x_i = \sum_{l=0}^{i-1} \frac{r^*_l - r^*_{l+1}}{c - r^*_l}.$$

Also,

$$g_{k,k}(\boldsymbol{x}, \boldsymbol{r}^*, \boldsymbol{b}^*, c) - g_{k,0}(\boldsymbol{x}, \boldsymbol{r}^*, \boldsymbol{b}^*, c) = -c + cx_k = 0$$

implies $x_k = 1$. Therefore c should be a root of

$$h(\boldsymbol{r}^*, c) := \sum_{i=0}^{k-1} \frac{r^*_i - r^*_{i+1}}{c - r^*_i} - 1 = 0.$$

Lemma 3. *The root of* $h(\boldsymbol{r}^*, c) = 0$ *uniquely exists between 1 and 2.*

We write the determined roots as $\boldsymbol{x}^*$ and c^*. Set $x^*_0 = 0$. Our guess turns out to be correct.

Lemma 4. $(\boldsymbol{x}^*, \boldsymbol{b}^*, c^*)$ *is a global optimum to problem* $(\mathcal{Q})$.

Next, we regard c^* as the optimal value function to $(\mathcal{Q})$ and formulate a problem $(\mathcal{R})$ with $\boldsymbol{r}^*$ being again a variable. Since problem $(\mathcal{R})$ is convex, a standard argument leads to an optimal solution.

$$(\mathcal{R}) \quad \text{minimize } c^* \\ \text{subject to } 1 = r_0 > r_1 > \cdots > r_k = 0.$$

Lemma 5. $\overline{\boldsymbol{r}}$ *in (4) is a global optimum to problem* $(\mathcal{R})$*. The optimal value is* $\overline{c}$ *in (7).*

We obtain $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{b}}$ in (3), (5), and (6) by substituting $\overline{\boldsymbol{r}}$ in $\boldsymbol{x}^*$ and $\boldsymbol{b}^*$. Lemmas 4 and 5 imply that $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{r}}, \overline{\boldsymbol{b}}, \overline{c})$ is a global optimum to problem $(\mathcal{P})$. □

4 Supremum of the Best Possible Competitive Ratio

We first see for each of the fixed-k and arbitrary-k cases that $\sup \tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ is equal to the matching upper and lower bound of the competitive ratio in the ordinary sense. In the literature of ski rental, c_u is said to be an upper bound if for all $(\boldsymbol{r}, \boldsymbol{b})$, there exists a strategy $\boldsymbol{x}$ which is c_u-competitive. It is observed that the set of such c_u is equivalent to the set of upper bounds of $\tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ over arbitrary instances. Hence, if one identifies $\sup \tilde{c}(\boldsymbol{r}, \boldsymbol{b})$, it is equal to the least upper bound. On the other hand, c_l is called a lower bound if there exists $(\boldsymbol{r}, \boldsymbol{b})$ for which any $\boldsymbol{x}$ is not c_l-competitive. Analogously, we have that $\sup \tilde{c}(\boldsymbol{r}, \boldsymbol{b})$ coincides with the greatest lower bound. We add that $c_l \leq \sup \tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \leq c_u$ holds accordingly.
Fixed k. We present the supremum each for $k = 2$ and $k = 3$, whose proofs are left to the full version.

Theorem 3. $\sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(2)\}$ *is the root of* $c^3 - 4c^2 + 5c - 3 = 0$*, which is approximately* 2.47*.*

Theorem 4. $\sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(3)\}$ *is the root of* $c^3 - 5c^2 + 8c - 5 = 0$*, which is approximately* 2.75*.*

Below is the instance which asymptotically achieves each supremum: For $k = 2$, $(\overline{r}_0, \overline{r}_1, \overline{r}_2) = (1, \varepsilon_1, 0)$ and $(\overline{b}_{0,1}, \overline{b}_{0,2}, \overline{b}_{1,2}) = (0.68, 1, 1)$. For $k = 3$, $(\overline{r}_0, \overline{r}_1, \overline{r}_2, \overline{r}_3) = (1, \varepsilon_1, \varepsilon_2, 0)$ and $(\overline{b}_{0,1}, \overline{b}_{0,2}, \overline{b}_{0,3}, \overline{b}_{1,2}, \overline{b}_{1,3}, \overline{b}_{2,3}) = (0.40, 0.70, 1, 0.70, 1, 1)$. The ε's are small positive numbers. See Figures 5 and 6. These instances are understood to be the hardest ones for the online player. Let us see the entries. The r_i's of the intermediate states all approach zero. Also, they are investment instances. Indeed, $\overline{b}_{0,2} = \overline{b}_{1,2}$ for $k = 2$, and $\overline{b}_{0,2} = \overline{b}_{1,2}$ and $\overline{b}_{0,3} = \overline{b}_{1,3} = \overline{b}_{2,3}$ for $k = 3$. From this observation we conjecture that for every k an investment instance achieves the supremum.

For each k, two different strategies asymptotically achieve the supremum. Note here that unlike the case of an additive instance, the choice of which states to skip affects the online player's cost. The strategies are: For $k = 2$, $\overline{\boldsymbol{x}}_1 = (0, 0.68, 0.68)$ with $0 \prec 2$ (i.e., skipping state 1) and $\overline{\boldsymbol{x}}_2 = (0, 0.47, 1/\delta_1)$ with $0 \prec 1 \prec 2$. And for $k = 3$, $\overline{\boldsymbol{x}}_1 = (0, 0.23, 1/\delta_2, 1/\delta_2)$ with $0 \prec 1 \prec 3$ and $\overline{\boldsymbol{x}}_2 = (0, 0.40, 0.40, 1/\delta_2)$ with $0 \prec 2 \prec 3$. δ's are small positive numbers determined by ε's in $\overline{\boldsymbol{r}}$. Figures 5 and 6 illustrate these strategies as well.
Arbitrary k. It seems difficult to obtain a supremum for arbitrary k in the same way as we have done with fixed k. Nevertheless, the interval where the supremum can exist is implied by the known upper and lower bounds of the competitive ratio in the standard sense.

Theorem 5 ([Dam03])**.** $(5 + \sqrt{5})/2 < \sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k), k \geq 2\}$.

Theorem 6 ([BCN00])**.** $\sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I(k), k \geq 2\} \leq 4$.

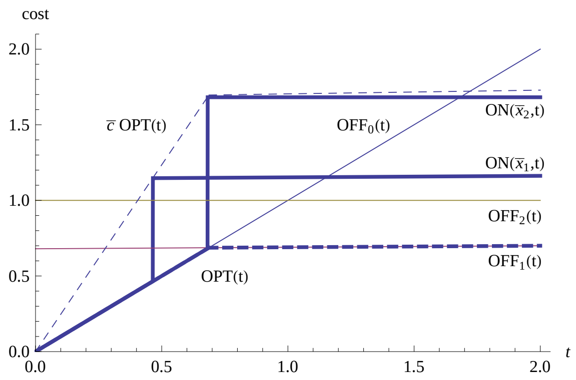

Figure 5: Instance and strategies that asymptotically achieve the supremum for $k = 2$. $\overline{r}_1$ is set 0.01.

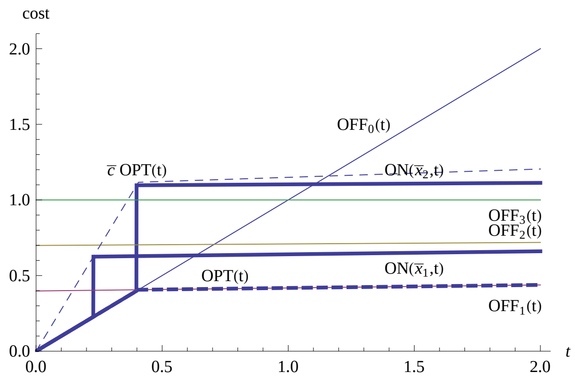

Figure 6: Instance and strategies that asymptotically achieve the supremum for $k = 3$. $\overline{r}_1$ and $\overline{r}_2$ are set 0.02 and 0.01, respectively.

5 Subclasses of the Multislope Ski Rental

Additive Instance. Theorem 2 and Corollary 2 provide the infimum for $I_A(k)$ as well, since Lemma 2 guarantees that an additive instance achieves the minimum.

We establish a supremum by combining with an existing result. We construct an instance for which any strategy cannot be $(2 - \varepsilon)$-competitive. The proof will be provided in the full version. On the other hand, Irani et al. [ISG03] provided the Lower Envelope strategy that is 2-competitive. Note that unlike the case of arbitrary instances, the supremum is constant regardless of k.

Theorem 7. *Fix $k \geq 2$. Given $0 < \varepsilon < 1$, let $(\boldsymbol{r}, \boldsymbol{b}) \in I_A(k)$ be such that $1 = r_0 > r_1 > \cdots > r_{k-1} = 1 - \varepsilon$, $r_k = 0$, $b_{0,i} = 1 - r_i$ for $0 \leq i \leq k$, and $b_{i,j} = b_{0,j} - b_{0,i}$ for $0 \leq i < j \leq k$. Then, for any ε, no strategy is $(2 - \varepsilon)$-competitive.*

Theorem 8 ([ISG03])**.** *Fix $k \geq 2$. For any instance $(\boldsymbol{r}, \boldsymbol{b}) \in I_A(k)$, there is a 2-competitive strategy.*

Corollary 3. *For any $k \geq 2$, $\sup\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I_A(k)\} = 2$.*

Investment Instance. We prove Theorem 9 which states that the infimum is independent of k. We will provide the proof in the full version.

As for the supremum, Theorems 3 and 4 hold true for $I_I(2)$ and $I_I(3)$, respectively, since an investment instance achieves each of the suprema. We mention that Theorem 6 was originally proved for $I_I(k)$.

Theorem 9. *For any $k \geq 2$, $\inf\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I_I(k)\} = \min\{\tilde{c}(\boldsymbol{r}, \boldsymbol{b}) \mid (\boldsymbol{r}, \boldsymbol{b}) \in I_I(k)\} = 2$. The strategy and the instance that achieve the minimum are: $\overline{x}_0 = 0$ and $\overline{x}_1 = \overline{x}_2 = \cdots = \overline{x}_k = 1$; $\overline{\boldsymbol{r}}$ is such that $1 = \overline{r}_0 > \overline{r}_1 > \cdots > \overline{r}_k = 0$; $\overline{b}_{i,j} = 1 - \overline{r}_j$ for $0 \leq i < j \leq k$.*

References

[AIS08] J. Augustine, S. Irani, and C. Swamy. Optimal power-down strategies. *SIAM J. Comput.*, 37(5):1499–1516, 2008.

[BCN00] Y. Bejerano, I. Cidon, and J. Naor. Dynamic session management for static and mobile users: a competitive on-line algorithmic approach. In *Proc. DIAL-M '00*, pages 65–74, 2000.

[Dam03] P. Damaschke. Nearly optimal strategies for special cases of on-line capital investment. *Theor. Comput. Sci.*, 302(1-3):35–44, 2003.

[EYKL99] R. El-Yaniv, R. Kaniel, and N. Linial. Competitive optimal on-line leasing. *Algorithmica*, 25(1):116–140, 1999.

[Fle01] R. Fleischer. On the Bahncard problem. *Theor. Comput. Sci.*, 268(1):161–174, 2001.

[ISG03] S. Irani, S. Shukla, and R. Gupta. Online strategies for dynamic power management in systems with multiple power-saving states. *ACM Trans. Embed. Comput. Syst.*, 2(3):325–346, 2003.

[Kar96] A. R. Karlin. On the performance of competitive algorithms in practice. In A. Fiat and G. J. Woeginger, editors, *Online Algorithms*, volume 1442 of *LNCS*, pages 373–384. Springer, 1996.

[KKR01] A. R. Karlin, C. Kenyon, and D. Randall. Dynamic TCP acknowledgement and other stories about $e/(e-1)$. In *Proc. STOC '01*, pages 502–509, 2001.

[KMMO94] A. R. Karlin, M. S. Manasse, L. McGeogh, and S. Owicki. Competitive randomized algorithms for nonuniform problems. *Algorithmica*, 11(6):542–571, 1994.

[KMRS88] A. R. Karlin, M. S. Manasse, L. Rudolph, and D. D. Sleator. Competitive snoopy caching. *Algorithmica*, 3:77–119, 1988.

[LPSR08a] Z. Lotker, B. Patt-Shamir, and D. Rawitz. Rent, lease or buy: Randomized algorithms for multislope ski rental. In *Proc. STACS '08*, pages 503–514, 2008.

[LPSR08b] Z. Lotker, B. Patt-Shamir, and D. Rawitz. Ski rental with two general options. *Inf. Process. Lett.*, 108(6):365–368, 2008.