Competitive Analysis for the 3-Slope Ski-Rental Problem with the Discount Rate

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Abstract

In the 3-slope ski-rental problem, the player is asked to determine a strategy, that is, (i) whether to buy a ski wear and then a ski set separately, or to buy them at once for a discount price, and (ii) when to buy these goods. If the player has not got any thing, he/she can rent it for some price. The objective is to minimize the total cost, under the assumption that the player does not know how many times he/she goes skiing in the future. We reveal that even with a large discount for buying at once available, there is some price setting for which to buy the goods separately is a more reasonable choice. We also show that the performance of the optimal strategy may become arbitrarily worse, when a large discount is offered.

1 Introduction

Suppose that the price of a ski wear is \$300, the price of a ski set, which is a pair of skis and a pair of boots, is \$500, and a combo of a ski wear and a ski set is offered for \$600. Then, buying the combo may seem nice. The analysis in this paper shows, however, that if the cost of the rent of a ski set is cheap, to buy the goods separately can be reasonable.

A price setting as above can be formulated as the 3-slope ski-rental problem [LPSR08]. The objective is to minimize the total cost, under the assumption that the player does not know how many times he/she goes skiing in the future. For this problem, there are two types of strategy:

- Rent both a ski wear and a ski set for some times, and then buy the combo of them (later called "type 02").
- Rent both a ski wear and a ski set for some times, and then buy a ski wear. After that, rent only a ski set for some times, and finally buy a ski set (later called type "012").

In this way, a strategy is characterized by the type and also, importantly, the number of times of rentals. Note that the player adopts a strategy as long as he/she keeps on skiing. For example, it happens that the player quits skiing before buying anything.

The standard performance measure of such a strategy is the competitive ratio [BE98], defined as the maximum ratio of the cost incurred by the strategy to the optimal offline cost, that is, the cost with the number of times of skiing known in advance. An optimal strategy for the 3-slope ski-rental problem is numerically calculated by the algorithm of Augustine et al. [AIS08]. Nevertheless, it has not been clear what instance admits a small competitive ratio and does not. In this paper we carry out a parametric analysis with the discount rate that indicates how cheap the combo is compared with the sum of the prices of the goods.

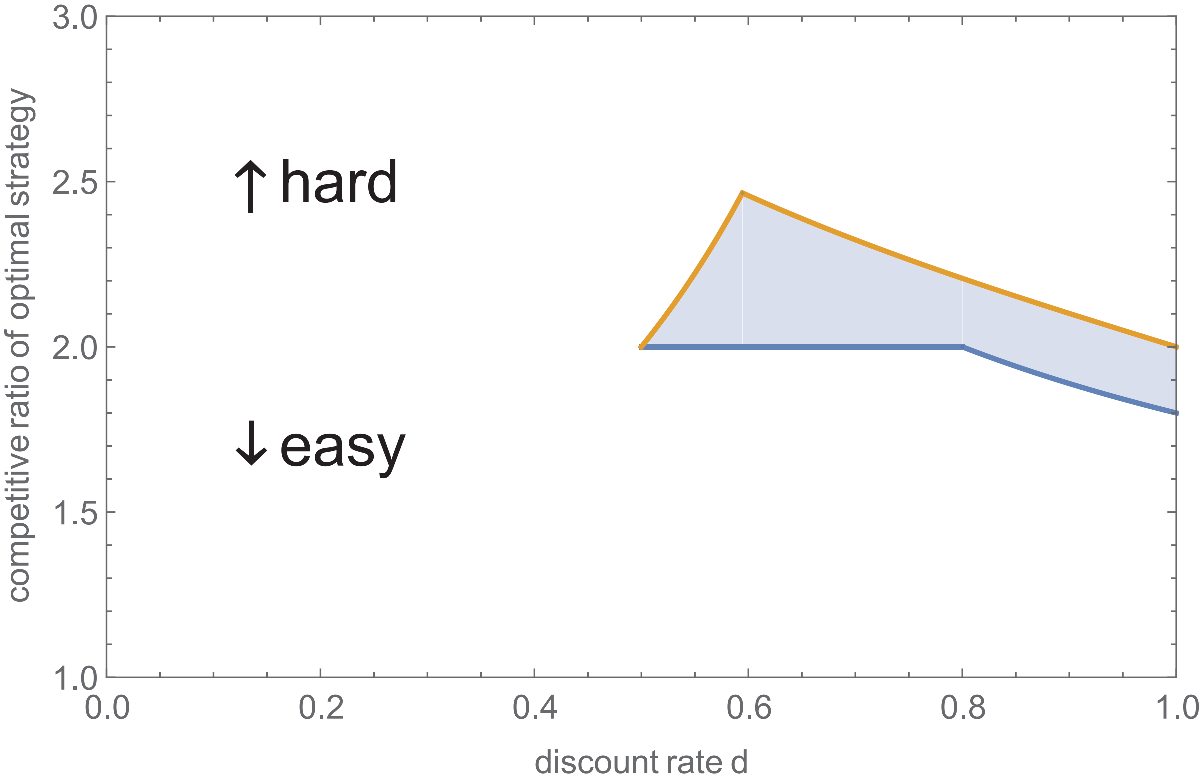

Figure 1: Range of the competitive ratio of the optimal strategy for instances in the narrow set $(= N_d)$ with each discount rate d. N_d is non-empty only when $\frac{1}{2} \leq d \leq 1$.

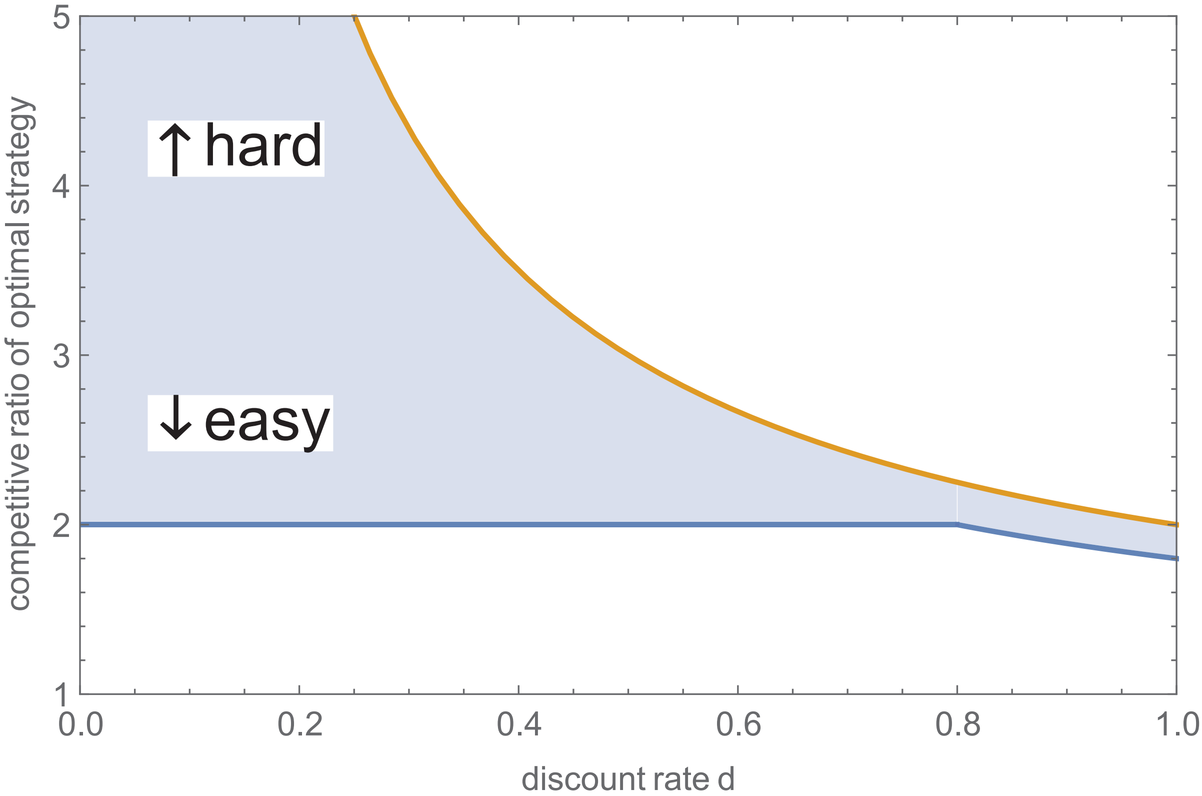

Figure 2: Range of the competitive ratio of the optimal strategy for instances in the wide set $(= W_d)$ with each discount rate d.

Table 1: Competitive ratio, denoted by CR, of the optimal strategy for the easiest/hardest instance each in for different sets.

set of instances	CR of optimal strategy for the easiest instance	CR of optimal strategy for the hardest instance
narrow set ($= N$)	$\frac{9}{5}$ [FKF14a]	$2.47 - \varepsilon$ [FKF14a]
narrow set with fixed discount rate ($= N_d$)	2, **if** $\frac{1}{2} \le d < \frac{4}{5}$; $1 + \frac{4}{5d}$, **if** $\frac{4}{5} \le d \le 1$	$\frac{1}{1-d} - \varepsilon$, **if** $\frac{1}{2} \le d < 0.59$; $\frac{1}{2}\left(3 + \sqrt{\frac{4}{d} - 3}\right) - \varepsilon$, **if** $0.59 \le d \le 1$
wide set ($= W$)	$\frac{9}{5}$	∞
wide set with fixed discount rate ($= W_d$)	2, **if** $0 \le d < \frac{4}{5}$; $1 + \frac{4}{5d}$, **if** $\frac{4}{5} \le d \le 1$	$1 + \frac{1}{d}$

1.1 Our Contribution

We derive an optimal strategy in an analytical form for an individual instance of the 3-slope ski-rental problem. We reveal that there is a price setting with a small discount rate for which to buy a ski wear and a ski set separately is better than to buy the combo.

Investigating the relation between the discount rate and the competitive ratio of the optimal strategy, we find the easiest instance and the hardest instance among those with a fixed discount rate. We consider two different sets of instances: a narrow set of instances originated from *Dynamic Power Management* [ISG03] and a wide one which represents the ski rental settings more naturally. Our result is summarized in Table 1.1. Figures 1 and 2 show the ranges of the competitive ratio of the optimal strategy for the instance sets with a fixed discount rate.

It is a natural intuition that when the discount rate is extremely large or small, the choice by the player is trivial and therefore the instance is easy. That is to say, if the discount rate is very small, to buy the combo is of course reasonable. If the discount rate is very large, to buy the goods one-by-one can avoid the risk of quitting skiing. The intuition is correct for the narrow set of instances; the instance is easy when the discount rate is extremely large or small. In contrast, it is not the case for the wide set of instances; there is a hard instance with the discount rate being almost zero. (See Figures 1 and 2.)

1.2 Application

We mention that the study on the multislope ski-rental problem [LPSR08], a general form of the 3-slope ski-rental problem, originally started as Dynamic Power Management on electric devices equipped with the "Sleep" mode [ISG03]. A strategy there is a schedule of automatically turning the device to "Sleep" and shutting it "Off" when there is no user response. The objective is to minimize the energy consumption during an idle period of uncertain length, including the energy consumption to resume the device. The 3-slope ski-rental problem and Dynamic Power Management correspond as the followings and thus turn out to be equivalent.

- the number of times of skiing = the length of the idle period.

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- buy the combo = keep staying state "On" for some time from the beginning of the idle period, and then shut the device "Off".
- buys a ski wear and a ski set separately = keep staying state "On" for some time from the beginning of the idle period, next turn the device to "Sleep" and wait for a while, and finally shut it "Off".
- the price of a ski wear = the total energy consumption for turning from "On" to "Sleep", and turning from "Sleep" to "On".
- the price of a ski set = the total energy consumption for turning from "Sleep" to "Off", and turning from "Off" to "On".
- the price of the combo = the total energy consumption for turning from "On" to "Off", and turning from "Off" to "On".
- the cost of renting both a ski wear and a ski set once = power consumed in "On".
- the cost of renting only a ski set once = power consumed in "Sleep".

This correspondence suggests further application in particular in decisions in management, such as management of factories where irregular and unscheduled orders arrive.

1.3 Related Work

The earliest idea of the ski-rental problem is found in the paper [KMRS88], where the goods to be bought/rented are entire ski gear. The question is when to buy them after renting them.

The k-slope ski-rental problem [LPSR08] is an extension that ski gear consist of $(k-1)$ parts. (The original model was that there are $(k-1)$ different areas options for getting entire ski gear.) The reason why the problem is called "k-slope" is that once the player chooses a strategy, the cost increases linearly with the number of times of skiing. The algorithm of Augustine et al. [AIS08] calculates an optimal strategy.

For arbitrary k, Damaschke [Dam03] gave a lower bound on the competitive ratio of $\frac{5+\sqrt{5}}{2} \approx 3.62$. An upper bound of 4 is immediately derived from the result by Bejerano et al. [BCN00].

For each $k \geq 3$, the paper [FKF14b] gave a lower bound for the k-slope ski-rental problem. A better upper bound than 4 due to [BCN00] is in general unknown.

2 Preliminaries

2.1 Problem Statement

We define an *instance* of the 3-slope ski-rental problem. Our notation basically follows the paper [FKF14a]. Although the indices may seem redundant, we believe that they will help when comparing results. In the general case an index corresponds to a state that the player has already bought some of ski gear. We denote an instance by $(r_1, b_{0,1}, b_{1,2})$ such that:

- the price of a ski wear = $b_{0,1}$,
- the price of a ski set (i.e., a pair of skis and a pair of boots) = $b_{1,2}$,
- the price of the combo (i.e., a ski wear and a ski set) = 1,
- the cost of renting both a ski wear and a ski set once = 1, and

- the cost of renting only a ski set once $= r_1$.

The prices are normalized so that the price of the combo $= 1$. Namely, $b_{0,1}$ and $b_{1,2}$ are regarded as ratios. Of course, $b_{0,1} + b_{1,2}$ can be larger than one.

For ease of calculation, we hereafter assume that the number of times of skiing and also the number of times of renting goods are fractional numbers. Results from our analysis can be converted in the real price setting by rescaling and rounding.

A *strategy of type 012* is written as $\boldsymbol{x} = (x_1, x_2)$, where

- the number of times of renting both a ski wear and a ski set $= x_1$, and
- the number of times of renting only a ski set $= x_2 - x_1$.

After x_2 times of skiing, the player buys a ski set for a cost of $b_{1,2}$. Note that the number of renting goods may not be realized when the player quits skiing earlier. We denote the number of times of skiing by t. Then, the total cost of the type-012 strategy is

$$ON(\boldsymbol{x}, t) := \begin{cases} t, & 0 \le t < x_1; \\ r_1(t - x_1) + x_1 + b_{0,1}, & x_1 \le t < x_2; \\ r_1(x_2 - x_1) + x_1 + b_{0,1} + b_{1,2}, & x_2 \le t. \end{cases}$$

The first expression shows the cost when the player keeps renting the both goods for t times. The second one is the cost when after renting the both goods for x_1 times, the player buys a ski wear for $b_{0,1}$, and then rents only a ski set for $(t - x_1)$ times. The last one is the cost when after renting the both goods for x_1 times, the player buys a ski wear for $b_{0,1}$, then the player rents only a ski set for $(x_2 - x_1)$ times, and finally buys a ski set for $b_{1,2}$.

We denote a *strategy of type 02* also by $\boldsymbol{x} = (x_1, x_2)$, where

- the number of times of renting both a ski wear and a ski set $= x_2$.

After x_2 times of skiing, the player buys both a ski wear and a ski set for a cost of one. We always set $x_1 := x_2$ for a strategy of type 02. The cost for skiing for t times is

$$ON(\boldsymbol{x}, t) := \begin{cases} t, & 0 \le t < x_2; \\ x_2 + 1, & x_2 \le t. \end{cases}$$

If the player knew t in advance, the best action would be one of the followings: (i) to keep renting both the goods, (ii) to buy only a ski wear at the beginning and then to keep renting only a ski set, or (iii) to buy the combo at the beginning. It is easily confirmed that any other choice requires more cost. By choosing the best of the three, the cost is

$$OFF(t) := \min \{t, r_1 t + b_{0,1}, 1\},$$

which we call the *optimal offline cost*.

For an instance such that $b_{0,1} > 1 - r_1$, the case (ii) is never chosen as the best option. Thus, the optimal offline cost is explicitly written as

$$OFF(t) = \begin{cases} t, & 0 \le t < 1; \\ 1, & 1 \le t. \end{cases}$$

In contrast, for an instance with $b_{0,1} \le 1 - r_1$, we have

$$OFF(t) = \begin{cases} t, & 0 \le t < \frac{b_{0,1}}{1-r_1}; \\ r_1 t + b_{0,1}, & \frac{b_{0,1}}{1-r_1} \le t < \frac{1-b_{0,1}}{r_1}; \\ 1, & \frac{1-b_{0,1}}{r_1} \le t. \end{cases}$$

We evaluate the performance of a strategy by the *competitive ratio*

$$R_{\boldsymbol{x}} = \sup_{t \geq 0} \frac{ON(\boldsymbol{x}, t)}{OFF(t)}.$$

Namely, this ratio indicates how much the strategy can be charged compared with the optimal offline cost. A small competitive ratio signifies a good performance.

For an instance $(r_1, b_{0,1}, b_{1,2})$, we say that strategy $\boldsymbol{x}$ is *optimal* if $R_{\boldsymbol{x}'} \geq R_{\boldsymbol{x}}$ for all $\boldsymbol{x}'$. Note that the optimality in this paper is always established for each individual instance and does not guarantee the performance for all instances.

Suppose that strategies $\boldsymbol{x}$ and $\boldsymbol{x}'$ are optimal for instances $(r_1, b_{0,1}, b_{1,2})$ and $(r'_1, b'_{0,1}, b'_{1,2})$, respectively. We say that $(r_1, b_{0,1}, b_{1,2})$ is *easier* (*harder*) than $(r'_1, b'_{0,1}, b'_{1,2})$, if $R_{\boldsymbol{x}} < R_{\boldsymbol{x}'}$ ($R_{\boldsymbol{x}} > R_{\boldsymbol{x}'}$, respectively) holds.

2.2 Set of Instances and the Discount Rate

We define sets of instances to be analyzed. The set of instances for the 3-slope ski-rental problem considered in [FKF14a] is

$$N := \{(r_1, b_{0,1}, b_{1,2}) \mid 0 < r_1 < 1, 0 \leq b_{0,1} \leq 1, 0 \leq b_{1,2} \leq 1, b_{0,1} + b_{1,2} \geq 1, b_{0,1} \leq 1 - r_1\}.$$

This is originally from Dynamic Power Management [ISG03]. It is known that the competitive ratio of the optimal strategy for the hardest instance in N is $2.47 - \varepsilon$, whereas that for the easiest instance is $\frac{9}{5} = 1.8$ [FKF14a]. (See Table 1.1.)

We define the *discount rate* d as the ratio of the total price of a ski wear and a ski set, to the price of the combo. In short, $d := \frac{1}{b_{0,1} + b_{1,2}}$. We consider a set of instances that have a fixed discount rate

$$N_d := \left\{ (r_1, b_{0,1}, b_{1,2}) \in N \;\middle|\; \frac{1}{b_{0,1} + b_{1,2}} = d \right\}.$$

For any $(r_1, b_{0,1}, b_{1,2}) \in N$, due to $1 \geq b_{1,2} = \frac{1}{d} - b_{0,1}$, we have $b_{0,1} \geq \frac{1}{d} - 1$. On the other hand, $b_{0,1} \leq 1$ is a constant. Besides, $b_{0,1} + b_{1,2} \geq 1$. Therefore, $\frac{1}{2} \leq d \leq 1$ is necessary for N_d to be non-empty.

There are some constraints in N which are originated in Dynamic Power Management: $b_{0,1} \leq 1$ and $b_{1,2} \leq 1$. According to the correspondence in Section 1.2, the former (latter) constraint means that the cost for resuming (shutting down) from "Sleep" is less than that for resuming (shutting down, respectively) from "Off". This is quite valid for electric devices. An instance with $b_{0,1} > 1$ or $b_{1,2} > 1$ may feel strange because this means that the combo is cheaper than a single item. However, it is occasionally the case that for example, a train ticket to a further destination is a bit cheaper than that to the exact destination. We are interested in what strategy is chosen through the competitive analysis for such a case.

In addition, the constraint $b_{0,1} \leq 1 - r_1$ is set to leave the possibility of choosing the case (ii) for the optimal offline cost in Section 2.1, which is irrelevant to the price settings.

In this paper we consider also the set of instances without these constraints

$$W = \{(r_1, b_{0,1}, b_{1,2}) \mid 0 < r_1 < 1, 0 \leq b_{0,1}, \\ 0 \leq b_{1,2}, b_{0,1} + b_{1,2} \geq 1\}.$$

Similarly to N_d, we deal with the set of instances with a fixed discount rate

$$W_d = \left\{ (r_1, b_{0,1}, b_{1,2}) \in W \;\middle|\; \frac{1}{b_{0,1} + b_{1,2}} = d \right\},$$

which is non-empty for any $0 < d \leq 1$.

3 Optimal Online Strategy

In this section we derive an optimal strategy for a given instance in the wide instance set W. We begin by a lemma for calculating the competitive ratio easily.

Lemma 1. *For any instance $(r_1, b_{0,1}, b_{1,2})$ in W and any strategy $\boldsymbol{x}$, it holds that*

$$R_{\boldsymbol{x}} = \max\left\{\frac{ON(\boldsymbol{x}, x_1)}{OFF(x_1)}, \frac{ON(\boldsymbol{x}, x_2)}{OFF(x_2)}\right\}.$$

Proof. Although in this proof we assume that $\boldsymbol{x} = (x_1, x_2)$ is of type 012 (i.e., $x_1 < x_2$), the argument immediately holds for a strategy of type 02 (i.e., $x_1 = x_2$) as well. By definition of ON and OFF, one can see that function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ is defined on each of the intervals that are divided by the points $t = 0$, x_1, x_2, $\frac{r_1}{1-b_{0,1}}$, 1, and $\frac{1-b_{0,1}}{r_1}$, though the order here may not be different from their actual order. We prove the lemma by claiming that the function attains a maximum at either x_1 or x_2, and does not attain a maximum elsewhere.

Since on each of the intervals, ON and OFF are linear functions, $\frac{ON(\boldsymbol{x},t)}{OFF(t)}$ has a form of a linear function divided by a linear function. Therefore, function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ is monotone on each interval. This leads us to that function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ never takes a maximum at an inner point of some interval. In other words, the function is maximized at one of the endpoints of the intervals.

We are going to confirm that function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ does not attain a maximum at any other endpoints than x_1 or x_2. We consider instances with $b_{0,1} \le 1 - r_1$ and those with $b_{0,1} > 1 - r_1$ separately. Recall that for an instance with $b_{0,1} \le 1 - r_1$, OFF is defined on intervals $[0, \frac{r_1}{1-b_{0,1}})$, $[\frac{r_1}{1-b_{0,1}}, \frac{1-b_{0,1}}{r_1})$, and $[\frac{1-b_{0,1}}{r_1}, \infty)$, whereas for an instance with $b_{0,1} > 1 - r_1$, OFF is defined on intervals $[0, 1)$ and $[1, \infty)$. (See Section 2.1.)

We first consider the case of an instance with $b_{0,1} \le 1 - r_1$. (i) We investigate the point $t = \frac{r_1}{1-b_{0,1}}$. (i-a) If $\frac{r_1}{1-b_{0,1}} \in (0, x_1)$, $\frac{ON(\boldsymbol{x},t)}{OFF(t)} = \frac{t}{r_1 t + b_{0,1}} = \frac{1}{r_1} - \frac{b_{0,1}}{r_1(r_1 t + b_{0,1})}$ on the right neighborhood, which is an increasing function. Thus, function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ does not attain a maximum at $t = \frac{r_1}{1-b_{0,1}}$, since there is another t for which the function takes a larger value. We below show that the function cannot be maximized in this way. (i-b) If $\frac{r_1}{1-b_{0,1}} \in [x_1, x_2)$, $\frac{ON(\boldsymbol{x},t)}{OFF(t)} = \frac{r_1(t-x_1)+x_1+b_{0,1}}{t} = r_1 + \frac{(1-r_1)x_1+b_{0,1}}{t}$ on the left neighborhood. Since this is a decreasing function, function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ does not attain a maximum at $t = \frac{r_1}{1-b_{0,1}}$. (i-c) If $\frac{r_1}{1-b_{0,1}} \in [x_2, \infty)$, ON is constant on the left neighborhood. On the other hand, OFF increases there. Hence, function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ is a decreasing function on the left neighborhood. This implies that the function does not have a maximum there. (ii) We check $t = \frac{1-b_{0,1}}{r_1}$ similarly. We just show that function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ is an increasing function on the right neighborhood or that the function is a decreasing function on the left neighborhood. (ii-a) If $\frac{1-b_{0,1}}{r_1} \in (0, x_1)$, $\frac{ON(\boldsymbol{x},t)}{OFF(t)} = \frac{t}{1}$ on the right neighborhood, which is an increasing function. (ii-b) If $\frac{1-b_{0,1}}{r_1} \in [x_1, x_2)$, $\frac{ON(\boldsymbol{x},t)}{OFF(t)} = \frac{r_1(t-x_1)+x_1+b_{0,1}}{1}$ on the right neighborhood, which is an increasing function. (ii-c) If $\frac{1-b_{0,1}}{r_1} \in [x_2, \infty)$, ON is constant on the left neighborhood, while OFF increases there. Hence, function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ is a decreasing function on the left neighborhood.

We next turn to the case of an instance with $b_{0,1} > 1 - r_1$. We are going to see that $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ does not attain a maximum at $t = 1$. If $1 \in (0, x_1)$, the claim follows from (ii-a).

If $1 \in [x_1, x_2)$, we are done similarly from (ii-b). If $1 \in [x_2, \infty)$, (ii-c) implies the claim. Thus, $t = 1$ cannot be a maximizer of function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$.

We add that for any instance, function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ cannot achieve a supremum as $t \to 0$. As long as we consider a strategy with $x_1 > 0$, we have $\frac{ON(\boldsymbol{x},t)}{OFF(t)} = 1$ around $t = 0$. This value cannot become a supremum since there is another t such that $\frac{ON(\boldsymbol{x},t)}{OFF(t)} > 1$. If $x_1 = 0$, we do not have to care since $\frac{ON(\boldsymbol{x},x_1)}{OFF(x_1)}$ is already a candidate of the maximizer.

From these arguments we conclude that function $t \mapsto \frac{ON(\boldsymbol{x},t)}{OFF(t)}$ is maximized at either $t = x_1$ or $t = x_2$. □

The following two lemmas provide an optimal strategy for a given instance in a closed form. Lemma 2 shows that for instances in W with $b_{0,1} > 1 - r_1$, the optimal strategy is expressed independently of the parameters of the instance. Lemma 3 is for the set of instances in W with $b_{0,1} \leq 1 - r_1$, which contains N is as a proper subset.

Lemma 2. *Suppose that an instance $(r_1, b_{0,1}, b_{1,2})$ in W satisfies $b_{0,1} > 1 - r_1$. Then, a type-02 strategy of $\boldsymbol{x} = (1, 1)$ with $R_{\boldsymbol{x}} = 2$ is optimal.*

Proof. We first find an optimal strategy of type 02. For any type-02 strategy $\boldsymbol{x}$, we have the competitive ratio

$$R_{\boldsymbol{x}} = \frac{ON(\boldsymbol{x}, x_2)}{OFF(x_2)} = \begin{cases} \frac{x_2+1}{x_2}, & 0 < x_2 < 1; \\ \frac{x_2+1}{1}, & 1 \leq x_2. \end{cases}$$

It is obvious that $R_{\boldsymbol{x}}$ is minimized when $x_2 = 1$. The minimum value is 2.

To complete the proof, it suffices to show $R_{\boldsymbol{x}} > 2$ for any type-012 strategy $\boldsymbol{x}$. Let $\boldsymbol{x}$ be a type-012 strategy. We get

$$\frac{ON(\boldsymbol{x}, x_1)}{OFF(x_1)} = \begin{cases} \frac{x_1+b_{0,1}}{x_1}, & 0 < x_1 < 1; \\ \frac{x_1+b_{0,1}}{1}, & 1 \leq x_1. \end{cases}$$

If $x_1 < b_{0,1}$ or $x_1 > 2 - b_{0,1}$, then we are done, since this implies $R_{\boldsymbol{x}} > 2$. In what follows we assume $b_{0,1} \leq x_1 \leq 2 - b_{0,1}$. To guarantee the existence of such x_1, we also assume $b_{0,1} \leq 1$. Now, if $0 < x_2 < 1$, we derive $\frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)} = r_1 + \frac{(1-r_1)x_1+b_{0,1}+b_{1,2}}{x_2} > r_1 + (1 - r_1)x_1 + b_{0,1} + b_{1,2} \geq r_1 + (1 - r_1)x_1 + 1 \geq r_1 + (1 - r_1)b_{0,1} + 1 \geq r_1 + (1 - r_1)^2 + 1 = r_1(1 - r_1) + 2 > 2$. We applied $x_1 \geq b_{0,1}$, $b_{0,1} > 1 - r_1$, and $b_{0,1} + b_{1,2} \geq 1$. If $1 \leq x_2$, by a similar derivation, we know $\frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)} = \frac{r_1(x_2-x_1)+x_1+b_{0,1}+b_{1,2}}{1} \geq r_1 + (1 - r_1)x_1 + b_{0,1} + b_{1,2} > 2$. These facts lead us to the conclusion that any type-012 strategy $\boldsymbol{x}$ yields $R_{\boldsymbol{x}} > 2$. □

Lemma 3. *Suppose that an instance $(r_1, b_{0,1}, b_{1,2})$ in W satisfies $b_{0,1} \leq 1 - r_1$. Then, $\min\Big\{1 + \frac{1-b_{0,1}}{r_1}, 1 + \frac{1-r_1}{b_{0,1}}, 1 + \frac{1}{2}\Big(b_{1,2} + \sqrt{b_{1,2}^2 + 4b_{0,1}(1 - r_1)}\Big)\Big\}$ is the competitive ratio of the optimal strategy for the instance. (i) When $1 + \frac{1-b_{0,1}}{r_1}$ is the minimum, a type-02 strategy of $\boldsymbol{x} = (\frac{1-b_{0,1}}{r_1}, \frac{1-b_{0,1}}{r_1})$ is optimal. (ii) When $1 + \frac{1-r_1}{b_{0,1}}$ is the minimum, a type-02 strategy of $\boldsymbol{x} = (\frac{b_{0,1}}{1-r_1}, \frac{b_{0,1}}{1-r_1})$ is optimal. (iii) When $1 + \frac{1}{2}\Big(b_{1,2} + \sqrt{b_{1,2}^2 + 4b_{0,1}(1 - r_1)}\Big)$ is the minimum, a type-012 strategy of $\boldsymbol{x} = \Big(\frac{-b_{1,2}+\sqrt{b_{1,2}^2+4b_{0,1}(1-r_1)}}{2(1-r_1)}, \frac{1-b_{0,1}}{r_1}\Big)$ is optimal.*

Proof. (I) We begin by finding an optimal strategy of type 02. For any type-02 strategy $\boldsymbol{x}$, we obtain

$$R_{\boldsymbol{x}} = \frac{ON(\boldsymbol{x}, x_2)}{OFF(x_2)} = \begin{cases} \frac{x_2+1}{x_2}, & 0 < x_2 < \frac{b_{0,1}}{1-r_1}; \\ \frac{1}{r_1} + \frac{r_1 - b_{0,1}}{r_1} \cdot \frac{1}{r_1 x_2 + b_{0,1}}, & \frac{b_{0,1}}{1-r_1} \le x_2 < \frac{1-b_{0,1}}{r_1}; \\ x_2 + 1, & \frac{1-b_{0,1}}{r_1} \le x_2. \end{cases}$$

We immediately have that $R_{\boldsymbol{x}}$ decreases on $(0, \frac{b_{0,1}}{1-r_1})$ and increases on $[\frac{1-b_{0,1}}{r_1}, \infty)$. What one should note is that on $[\frac{b_{0,1}}{1-r_1}, \frac{1-b_{0,1}}{r_1})$, $R_{\boldsymbol{x}}$ is a decreasing function if $r_1 > b_{0,1}$ and is a non-decreasing function otherwise. Therefore, if $r_1 > b_{0,1}$, $R_{\boldsymbol{x}}$ attains a minimum of $1 + \frac{1-b_{0,1}}{r_1}$ at $x_2 = \frac{1-b_{0,1}}{r_1}$. Otherwise, $R_{\boldsymbol{x}}$ attains a minimum of $1 + \frac{1-r_1}{b_{0,1}}$ at $x_2 = \frac{b_{0,1}}{1-r_1}$.

(II) We next find an optimal strategy of type 012. For later use, we here write

$$\frac{ON(\boldsymbol{x}, x_1)}{OFF(x_1)} = \begin{cases} \frac{x_1 + b_{0,1}}{x_1}, & 0 < x_1 < \frac{b_{0,1}}{1-r_1}; \\ \frac{x_1 + b_{0,1}}{r_1 x_1 + b_{0,1}}, & \frac{b_{0,1}}{1-r_1} \le x_1 < \frac{1-b_{0,1}}{r_1}; \\ x_1 + b_{0,1}, & \frac{1-b_{0,1}}{r_1} \le x_1. \end{cases}$$

We are going to bound $\frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$ from below. Assuming $x_2 > x_1$, we express

$$\frac{ON(\boldsymbol{x}, x_2)}{OFF(x_2)} = \begin{cases} r_1 + \frac{(1-r_1)x_1 + b_{0,1} + b_{1,2}}{x_2}, & 0 < x_2 < \frac{b_{0,1}}{1-r_1}; \\ 1 + \frac{(1-r_1)x_1 + b_{1,2}}{r_1 x_2 + b_{0,1}}, & \frac{b_{0,1}}{1-r_1} \le x_2 < \frac{1-b_{0,1}}{r_1}; \\ r_1 x_2 + \frac{(1-r_1)x_1 + b_{0,1} + b_{1,2}}{1}, & \frac{1-b_{0,1}}{r_1} \le x_2. \end{cases}$$

With respect to x_2, $\frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$ decreases on $(0, \frac{b_{0,1}}{1-r_1})$ and $[\frac{b_{0,1}}{1-r_1}, \frac{1-b_{0,1}}{r_1})$, whereas $\frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$ increases on $[\frac{1-b_{0,1}}{r_1}, \infty)$. We thus obtain

$$\inf_{x_2} \frac{ON(\boldsymbol{x}, x_2)}{OFF(x_2)} = \begin{cases} 1 + (1-r_1)x_1 + b_{1,2}, & 0 \le x_1 < \frac{1-b_{0,1}}{r_1}; \\ x_1 + b_{0,1} + b_{1,2}, & \frac{1-b_{0,1}}{r_1} \le x_1. \end{cases}$$

The former is achieved when $x_2 = \frac{1-b_{0,1}}{r_1}$, and the latter is achieved when $x_2 \to x_1$.

In the rest of the proof, instead of minimizing $\max\{\frac{ON(\boldsymbol{x},x_1)}{OFF(x_1)}, \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}\}$, we minimize $f(x_1) := \max\{\frac{ON(\boldsymbol{x},x_1)}{OFF(x_1)}$, $\inf_{x_2} \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}\}$ with respect to x_1. $\inf_{x_2} \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$ is an increasing function on $(0, \infty)$. For $x_1 = \frac{b_{0,1}}{1-r_1}$, we have $\inf_{x_2} \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)} = 1 + b_{0,1} + b_{1,2} \ge 2$. On the other hand, for $x_1 = \frac{b_{0,1}}{1-r_1}$, $\frac{ON(\boldsymbol{x},x_1)}{OFF(x_1)} = 2 - r_1 < 2$. Hence, it holds that for all $x_1 \ge \frac{b_{0,1}}{1-r_1}$, $f(x_1) \ge 1 + b_{0,1} + b_{1,2}$, which implies that f attains a minimum anywhere on $(0, \frac{b_{0,1}}{1-r_1}]$. We now see function f for $x_1 \in (0, \frac{b_{0,1}}{1-r_1})$. $\frac{ON(\boldsymbol{x},x_1)}{OFF(x_1)}$ is a decreasing function and diverges to infinity as $x_1 \to 0$. In contrast, $\inf_{x_2} \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$ increases monotonically. We have already seen that for $x_1 = \frac{b_{0,1}}{1-r_1}$, $\frac{ON(\boldsymbol{x},x_1)}{OFF(x_1)} < \inf_{x_2} \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$. Therefore, there is a unique $x_1 \in (0, \frac{b_{0,1}}{1-r_1})$ such that $\frac{ON(\boldsymbol{x},x_1)}{OFF(x_1)} = \inf_{x_2} \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$. Together with the argument above, f attains a minimum at this point. We calculate this point as $x_1 = \frac{-b_{1,2} + \sqrt{b_{1,2}^2 + 4b_{0,1}(1-r_1)}}{2(1-r_1)}$. Then, the value of f is $1 + \frac{1}{2}\left(b_{1,2} + \sqrt{b_{1,2}^2 + 4b_{0,1}(1-r_1)}\right)$. From the above analysis on $\inf_{x_2} \frac{ON(\boldsymbol{x},x_2)}{OFF(x_2)}$, we have the value of $x_2 = \frac{1-b_{0,1}}{r_1}$. □

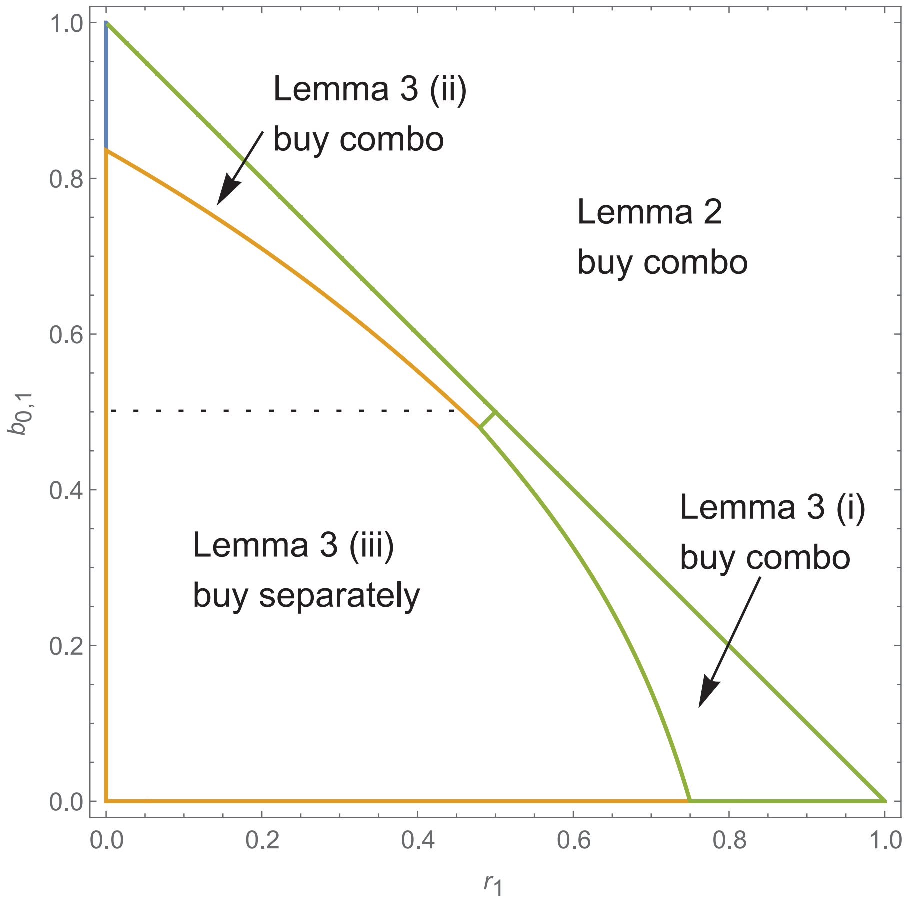

Figure 3: The corresponding lemma and the type of the optimal strategy for each instance $(r_1, b_{0,1}, 1.33 - b_{0,1})$ in $W_{0.75}$. (Note that $\frac{1}{0.75} \approx 1.33$.) The dotted line segment is $b_{0,1} = 0.5$.

To apply Lemmas 2 and 3 to instances having a fixed discount rate helps to understand their meanings. Figure 3 illustrates that each instance in $W_{0.75}$ applies which lemma and which case, and leads to what strategy. The example of prices given at the beginning of Section 1 is normalized to $b_{0,1} = 0.5$, $b_{1,2} = 0.83$, and $d = 0.75$, which is indeed an instance in $W_{0.75}$. Although the cost of the rent is not mentioned there, this instance corresponds to some point which satisfies $b_{0,1} = 0.5$ on Figure 3. One can observe that a point on $b_{0,1} = 0.5$ is contained in the region in which a strategy of type 012 optimal, if $0 < r_1 < 0.46$. This implies that if the cost of renting only a ski set is cheaper by 46% than the cost of renting both a ski wear and a ski set, then to buy the goods separately is reasonable.

The following corollary says that even for any small discount rate, which means that the combo is very cheap, there is a cost setting for the rent for which to buy the goods separately is better than to buy the combo. The corollary follows from the fact that as $r_1 \to 0$ and $b_{0,1} \to 0$, both $1 + \frac{1-b_{0,1}}{r_1}$ and $1 + \frac{1-r_1}{b_{0,1}}$ diverge to infinity, whereas $1 + \frac{1}{2}\Big(b_{1,2} + \sqrt{b_{1,2}^2 + 4b_{0,1}(1 - r_1)}\Big)$ remains finite.

Corollary 1. *For all $0 < d \le 1$, there is an instance in W_d for which a strategy of type 012 is optimal.*

In this way, fixing the discount rate is helpful for clarifying the relation between an instance and the competitive ratio of the optimal strategy. In the next section we find the easiest instance and the hardest instance with the discount rate being as a parameter.

4 Parametric Analysis with the Discount Rate

4.1 The Easiest Instance

We find an instance in W_d for which there is a strategy $\boldsymbol{x}$ that minimizes $R_{\boldsymbol{x}}$. The reason why we first consider W_d is that the easiest instance is eventually found in $N_d \subset W_d$ and therefore a similar result is established also for N_d.

Theorem 1. *(a) Suppose that d is a constant with $0 < d < \frac{4}{5}$. For any instance $(r_1, b_{0,1}, b_{1,2})$ in W_d and for any strategy $\boldsymbol{x}$, it follows that $R_{\boldsymbol{x}} \ge 2$. The equality holds when $(r_1, b_{0,1}, b_{1,2})$ is an arbitrary instance in W_d that satisfies $b_{0,1} \ge 1 - r_1$, and $\boldsymbol{x} = (1, 1)$. (b) Suppose that d is a constant with $\frac{4}{5} \le d \le 1$. For any instance $(r_1, b_{0,1}, b_{1,2})$ in W_d and for any strategy $\boldsymbol{x}$, it follows that $R_{\boldsymbol{x}} \ge 1 + \frac{4}{5d}$. The equality holds when $(r_1, b_{0,1}, b_{1,2}) = \big(1 - \frac{2}{5d}, \frac{2}{5d}, \frac{3}{5d}\big)$ and $\boldsymbol{x} = \big(\frac{1}{2}, 1\big)$.*

Proof. Lemma 2 implies that for any instance in W_d that satisfies $b_{0,1} > 1 - r_1$, strategy $\boldsymbol{x} = (1, 1)$ is optimal and $R_{\boldsymbol{x}} = 2$.

In the following we investigate instances in W_d such that $b_{0,1} \le 1 - r_1$, and see what instance admits a competitive ratio smaller than two. For simplicity of calculation, let $a = \frac{1}{d}$ $(1 \le a)$. Then, we have $b_{1,2} = a - b_{0,1}$. For an instance that applies (i) or (ii) of Lemma 3, the competitive ratio of the optimal strategy is at least two, due to $1 + \frac{1-b_{0,1}}{r_1} \ge 2$ and $1 + \frac{1-r_1}{b_{0,1}} \ge 2$. Thus, we focus on instances that apply (iii) of Lemma 3, which is the case where the last operand of $\min\Big\{1 + \frac{1-b_{0,1}}{r_1}, 1 + \frac{1-r_1}{b_{0,1}}, 1 + \frac{1}{2}\Big(a - b_{0,1} + \sqrt{(a - b_{0,1})^2 + 4b_{0,1}(1 - r_1)}\Big)\Big\}$ is the minimum. Let $h(r, b) := 1 + \frac{1}{2}\Big(a - b + \sqrt{(a - b)^2 + 4b(1 - r)}\Big)$. We are going to see how low the value of $h(r_1, b_{0,1})$ becomes by choosing an instance $(r_1, b_{0,1}, a - b_{0,1})$ in W_d. Since function h is monotonically non-decreasing with r, $h(r_1, b_{0,1}) \ge h(1 - b_{0,1}, b_{0,1}) = 1 + \frac{1}{2}\Big(a - b_{0,1} + \sqrt{(a - b_{0,1})^2 + 4b_{0,1}^2}\Big)$ holds for every instance. We minimize this with respect to $b_{0,1}$. By differentiating $h(1 - b, b)$ with b, we obtain $\frac{1}{2}\Big(-1 + \frac{-a+5b}{\sqrt{(a-b)^2+4b^2}}\Big)$, whose sign changes from plus to minus at $b = \frac{2}{5}a$ as

b increases. Hence, $h(r_1, b_{0,1})$ is minimized when $b_{0,1} = \frac{2}{5}a$ and $r_1 = 1 - b_{0,1} = 1 - \frac{2}{5}a$. The minimized value is $1 + \frac{4}{5}a$, which does not exceed two for $1 \le a \le \frac{5}{4}$.

What we can know from the analysis is that for W_d with $1 \le a \le \frac{5}{4}$, instance $(r_1, b_{0,1}, b_{1,2}) = \left(1 - \frac{2}{5}a, \frac{2}{5}a, \frac{3}{5}a\right)$ minimizes the competitive ratio of the optimal strategy. Lemma 3 says that the optimal strategy is a strategy of type 012 $\boldsymbol{x} = \left(\frac{-b_{1,2}+\sqrt{b_{1,2}^2+4b_{0,1}(1-r_1)}}{2(1-r_1)}, \frac{1-b_{0,1}}{r_1}\right) = \left(\frac{1}{2}, 1\right)$.

On the other hand, for W_d with $\frac{5}{4} < a$, any instance with $b_{0,1} > 1 - r_1$ minimizes the competitive ratio of the optimal strategy to two, as we have mentioned at the beginning of the proof. A strategy of type 02 of $\boldsymbol{x} = (1, 1)$ is optimal.

We add that for W_d with $\frac{5}{4} < a$, instances with $b_{0,1} = 1 - r_1$, which applies (i) or (ii) of Lemma 3, also admits a competitive ratio of two. The optimal strategy is $\boldsymbol{x} = (1, 1)$.

The statement of the theorem is obtained by replacing a with $\frac{1}{d}$. $\square$

The instance appearing in the statement (b) of Theorem 1 belongs to N_d. In the statement (a), we can choose an instance in N_d. Thus, the following corollary immediately follows.

Corollary 2. *(a) Suppose that d is a constant with $\frac{1}{2} \le d < \frac{4}{5}$. For any instance $(r_1, b_{0,1}, b_{1,2})$ in N_d and for any strategy $\boldsymbol{x}$, it follows that $R_{\boldsymbol{x}} \ge 2$. The equality holds when $(r_1, b_{0,1}, b_{1,2})$ is an arbitrary instance in N_d that satisfies $b_{0,1} = 1 - r_1$, and $\boldsymbol{x} = (1, 1)$. (b) Suppose that d is a constant with $\frac{4}{5} \le d \le 1$. For any instance $(r_1, b_{0,1}, b_{1,2})$ in N_d and for any strategy $\boldsymbol{x}$, it follows that $R_{\boldsymbol{x}} \ge 1 + \frac{4}{5d}$. The equality holds when $(r_1, b_{0,1}, b_{1,2}) = \left(1 - \frac{2}{5d}, \frac{2}{5d}, \frac{3}{5d}\right)$ and $\boldsymbol{x} = \left(\frac{1}{2}, 1\right)$.*

4.2 The Hardest Instance

It turns out that there is no hardest instance in N_d; the hardest instance is expressed as a limit. Thus, under the assumption that an optimal strategy is always chosen for the given instance, we analyze the least upper bound on its competitive ratio. In turn, an optimal strategy with finite x_1 and x_2 may not exist for some extreme instance. Therefore, we evaluate the infimum on the competitive ratio. For an instance $(r_1, b_{0,1}, b_{1,2})$, let $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}}$ be the maximum value of u such that $R_{\boldsymbol{x}'} \ge u$ for all $\boldsymbol{x}'$.

Theorem 2. *Let d_0 (≈ 0.594414) be the inverse of the unique real root of the equation $z^3 - 3z^2 + 4z - 3 = 0$. (i) Suppose that d is a constant with $\frac{1}{2} \le d < d_0$. The supremum value of $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}}$ for $(r_1, b_{0,1}, b_{1,2}) \in N_d$ is $\frac{1}{1-d}$, when $r_1 \to 0$, $b_{0,1} = \frac{1}{d} - 1$, and $b_{1,2} = 1$. Then, the infimum of $R_{\boldsymbol{x}}$ is achieved when $x_1 \to \frac{1}{d} - 1$ and $x_2 \to \frac{1}{d} - 1$. (ii) Suppose that d is a constant with $d_0 \le d < 1$. The supremum value of $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}}$ for $(r_1, b_{0,1}, b_{1,2}) \in N_d$ is $\frac{1}{2}\left(3 + \sqrt{\frac{4}{d} - 3}\right)$, when $r_1 \to 0$, $b_{0,1} = \frac{1}{d} - 1$, and $b_{1,2} = 1$. Then, the infimum of $R_{\boldsymbol{x}}$ is achieved when $x_1 \to \frac{1}{2}\left(-1 + \sqrt{\frac{4}{d} - 3}\right)$ and $x_2 \to \infty$. (iii) The supremum value of $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}}$ for $(r_1, b_{0,1}, b_{1,2}) \in N_1$ is 2, when (iii-a) $0 < r_1 < 1$, $b_{0,1} = 0$, and $b_{1,2} = 1$, or (iii-b) $r_1 \to 0$, $0 \le b_{0,1} \le 1$, and $b_{1,2} = 1 - b_{0,1}$. For (iii-a), the infimum of $R_{\boldsymbol{x}}$ is achieved when $x_1 = 0$ and $x_2 = \frac{1}{r_1}$. For (iii-b), the infimum of $R_{\boldsymbol{x}}$ is achieved when $x_1 \to b_{0,1}$ and $x_2 \to \infty$.*

For W_d, we can find the hardest instance. However, the optimal strategy is obtained as a limit.

Theorem 3. *(i) Suppose that d is a constant with $0 < d < 1$. The supremum value of $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}}$ for $(r_1, b_{0,1}, b_{1,2}) \in W_d$ is $1 + \frac{1}{d}$, when $0 < r_1 \le d$, $b_{0,1} = 0$, and $b_{1,2} = \frac{1}{d}$. Then, the infimum of $R_{\boldsymbol{x}}$ is achieved when $x_1 = 0$ and $x_2 \to \infty$. (ii) The supremum value of $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}}$ for $(r_1, b_{0,1}, b_{1,2}) \in W_1$ is 2, when (ii-a) $0 < r_1 < 1$, $b_{0,1} = 0$, and $b_{1,2} = 1$, or (ii-b) $r_1 \to 0$, $0 \le b_{0,1} \le 1$, and $b_{1,2} = 1 - b_{0,1}$. For (ii-a), the infimum of $R_{\boldsymbol{x}}$ is achieved when $x_1 = 0$ and $x_2 = \frac{1}{r_1}$. For (ii-b), the infimum of $R_{\boldsymbol{x}}$ is achieved when $x_1 \to b_{0,1}$ and $x_2 \to \infty$.*

We will give the proofs of Theorems 2 and 3 later. We have the next corollary on W from Theorems 1 and 3. In short, even for the optimal strategy, there is an instance in W for which the competitive ratio is arbitrarily large. This is caused by an instance where the price of a ski wear is almost zero, the price of a ski set is by far higher than that of the combo, and the cost of renting a ski set is very cheap, as the instance in Theorem 3. For such an instance, the optimal type-012 strategy is better than any type-02 strategy. Nevertheless, the competitive ratio of the optimal type-012 strategy is still large.

Corollary 3. *For all $M \geq \frac{9}{5}$, there is an instance in W for which the competitive ratio of any strategy is at least M.*

We here state a lemma which will be used in the proofs of Theorems 2 and 3. In particular, the hardest instance in Theorem 2 is obtained by letting r_1 approach zero. If we allow r_1 to be zero, then, in turn, the optimal strategy cannot be defined. The aim of the lemma is to carry out maximization on a domain that permits $r_1 = 0$.

Lemma 4. *Let a be a constant ≥ 1, and a_0 (≈ 1.68233) be the unique real root of the equation $z^3 - 3z^2 + 4z - 3 = 0$. Let $g(r, b) := \min\left\{1 + \frac{1-r}{b}, 1 + \frac{1-b}{r}, 1 + \frac{1}{2}\left(a - b + \sqrt{(a-b)^2 + 4b(1-r)}\right)\right\}$, where an operand is dropped if it is not defined. Then, function g attains a maximum as follows: (A) On domain $D_A := \{(r, b) \mid 0 \leq r \leq 1, a - 1 \leq b \leq 1 - r\}$. (A-i) If $a_0 \leq a < 2$, the maximum value is $\frac{a}{a-1}$, when $(r, b) = (0, a - 1)$. (A-ii) If $1 < a \leq a_0$, the maximum value is $\frac{1}{2}(3 + \sqrt{4a-3})$, when $(r, b) = (0, a - 1)$. (A-iii) If $a = 1$, the maximum value is 2, when $(r, b) = (y, 0)$ with $0 \leq y \leq 1$ or $(r, b) = (0, w)$ with $0 \leq w \leq 1$. (B) On domain $D_B := \{(r, b) \mid 0 \leq r \leq 1, 0 \leq b \leq 1 - r\} \supseteq D_A$. (B-i) If $1 < a$, the maximum value is $1 + a$, when $(r, b) = (y, 0)$ with $0 \leq y \leq \frac{1}{a}$. (B-ii) If $a = 1$, the maximum value is 2, when $(r, b) = (y, 0)$ with $0 \leq y \leq 1$ or $(r, b) = (0, w)$ with $0 \leq w \leq 1$.*

Proof. The first and second operands of the minimum operation in function g are non-increasing functions each with r and b. We have to take a bit care about the third one. Let $h(r, b) := 1 + \frac{1}{2}(a - b + \sqrt{(a-b)^2 + 4b(1-r)})$, as the same as the proof of Theorem 1. It is immediately observed that function h decreases or remains constant as r grows. Thus, $h(r, b) \leq h(0, b)$ holds for all r with $0 \leq r \leq 1$. We derive for all b with $0 \leq b \leq 1$, $\frac{\partial h(0,b)}{\partial b} = \frac{1}{2}\left(\frac{2-(a-b)}{\sqrt{(a-b)^2+4b}} - 1\right) = \frac{2(1-a)}{\left(\sqrt{(a-b)^2+4b}+2-(a-b)\right)\sqrt{(a-b)^2+4b}} \leq 0$, since we have $\sqrt{(a-b)^2 + 4b} + 2 - (a - b) \geq a - b + 2 - (a - b) > 0$ due to $a - b \geq 0$. Hence, $h(0, b)$ is non-increasing with b. Therefore, it holds that for all $(r, b) \in D_A$, $h(r, b) \leq h(0, a - 1)$. Similarly, for all $(r, b) \in D_B$, $h(r, b) \leq h(0, 0)$ holds.

Now, note that if $r \leq b$ and $r \leq 1 - b$, then $1 + \frac{1-r}{b} \leq 1 + \frac{1-b}{r}$. From this, the maximum value of function g for each of the domains is obtained as followings.

(A) For domain D_A, each operand of the minimum operation of g is maximized when $(r, b) = (0, a - 1)$. Thus, g attains a maximum of $\min\{1 + \frac{1}{a-1}, \frac{1}{2}\left(3 + \sqrt{4a-3}\right)\}$ at $(r, b) = (0, a - 1)$. Regard each operand as a function of a. As a grows, the former decreases, while the latter increases. These two operands take the same value, when $a = a_0$ (≈ 1.68233), where a_0 is the unique real root of the equation $z^3 - 3z^2 + 4z - 3 = 0$.

(A-i) For $a_0 \leq a \leq 2$, the maximum value of g is $g(0, a - 1) = 1 + \frac{1}{a-1}$. Since $1 + \frac{1-r}{b}$ is a decreasing function each with r and b, $(r, b) = (0, a - 1)$ is a unique maximizer of g.

(A-ii) For $1 < a < a_0$, the maximum value is $g(0, a - 1) = h(0, a - 1) = \frac{1}{2}\left(3 + \sqrt{4a-3}\right)$. Since $\frac{\partial h(0,b)}{\partial b} = \frac{1}{2}\left(\frac{1}{\sqrt{4a-3}} - 1\right) < 0$ for $b = a - 1$, and $h(r, a - 1) = \frac{1}{2}(3 + \sqrt{1 + 4(a-1)(1-r)})$ is a decreasing function with r, $(r, b) = (0, a - 1)$ is a unique maximizer.

(A-iii) For $a = 1$, the maximum value is $g(0,0) = h(0,0) = 2$. We have also $h(r,0) = h(0,b) = 2$ for $0 \le r \le 1$ and $0 \le b \le 1$. Since for $b > 0$, h is a decreasing function with r, any of $(r,b) = (y,0)$ with $0 \le y \le 1$ or $(r,b) = (0,w)$ with $0 \le w \le 1$ achieves the maximum.

(B) For domain D_B, each operand of the minimum operation of g is maximized when $(r,b) = (0,0)$. The first and second operands are not defined, since they diverge to infinity as $r \to 0$ and $b \to 0$. Thus, the maximum value of g is $g(0,0) = h(0,0) = 1 + a$.

(B-i) For $1 < a$, $(r,b) = (0,0)$ is not a unique maximizer, since $\frac{\partial h(0,b)}{\partial b} = -1 + \frac{1}{a} < 0$ for $b = 0$ but $h(r,0) = 1 + a$. We solve the equation $1 + \frac{1-b}{r} = 1 + \frac{1}{2}(a - b + \sqrt{(a-b)^2 + 4b(1-r)})$ with $b = 0$, and then get a root $(r,b) = (\frac{1}{a}, 0)$. This leads us to that for (r,b) with $r > \frac{1}{a}$ and $b = 0$, $1 + \frac{1-b}{r}$ is the minimum in the minimum operation of g. Note that now that $r > b$ and $r \le 1 - b$, $1 + \frac{1-r}{b} > 1 + \frac{1-b}{r}$ holds. Therefore, every $(r,b) = (y,0)$ with $0 \le y \le \frac{1}{a}$ is a maximizer of g.

(B-ii) For $a = 1$, the maximum value is $g(0,0) = h(0,0) = 2$. Similarly to (A-iii), we know that any of $(r,b) = (y,0)$ with $0 \le y \le 1$ or $(r,b) = (0,w)$ with $0 \le w \le 1$ achieves the maximum. □

Proof of Theorem 2. For ease of calculation, let $a = \frac{1}{d}$ $(1 \le a \le 2)$, as the same as the proof of Theorem 1. Again, we have $b_{1,2} = a - b_{0,1}$. Applying Lemma 3 to an instance in N_d with $b_{0,1} \le 1 - r_1$, we know that $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}} = \min\left\{1 + \frac{1-b_{0,1}}{r_1}, 1 + \frac{1-r_1}{b_{0,1}}, 1 + \frac{1}{2}\left(a - b_{0,1} + \sqrt{(a-b_{0,1})^2 + 4b_{0,1}(1-r_1)}\right)\right\}$. We are going to seek an instance that maximizes this. For $(r_1, b_{0,1}, b_{1,2}) \in N_d$, $1 \ge b_{1,2} = a - b_{0,1}$ holds. So, the range of $b_{0,1}$ is $a - 1 \le b_{0,1} \le 1$. If we allow r_1 to be chosen from the range $0 \le r_1 \le 1$, Lemma 4 (A) says that the maximum value is determined according to the range of a. However, the maximum is achieved by $(r_1, b_{0,1}, b_{1,2})$ which does not belong to N_d for cases (A-i), (A-ii), and some of (A-iii). Function g of Lemma 4 is continuous with respect to both r and b on D_A. Hence, for all $\varepsilon > 0$, there is $(r_1, b_{0,1}, b_{1,2})$ in N_d such that $g(r_1, b_{0,1}) >$ (maximum value of g) $- \varepsilon$.

When (A-i) in Lemma 4 is the case, $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}} = 1 + \frac{1-r_1}{b_{0,1}}$. Using (ii) of Lemma 3, we know that a type-02 strategy is optimal. When $r_1 \to 0$ and $b_{0,1} = a - 1$, we have $x_1 \to a - 1$ and $x_2 \to a - 1$.

When (A-ii) in Lemma 4 is the case, $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}} = 1 + \frac{1}{2}\left(a - b_{0,1} + \sqrt{(a-b_{0,1})^2 + 4b_{0,1}(1-r_1)}\right)$. By (iii) of Lemma 3, a type-012 strategy turns out be optimal. When $r_1 \to 0$ and $b_{0,1} = a - 1$, we have $x_1 \to \frac{1}{2}\left(-1 + \sqrt{\frac{4}{d} - 3}\right)$ and $x_2 \to \infty$.

When (A-iii) in Lemma 4 is the case, we can obtain a supremum as the same as (A-ii) above. We get two sets of instances that achieve the supremum. The optimal strategy is derived by applying (iii) of Lemma 3 for each.

The statement of the theorem is obtained by replacing a with $\frac{1}{d}$. □

Proof of Theorem 3. Lemma 2 states that for any instance in W_d with $b_{0,1} > 1 - r_1$, there is a strategy with a competitive ratio of two. In contrast, it is revealed that $\inf_{\boldsymbol{x}} R_{\boldsymbol{x}}$ achieves a supremum larger than two for those with $b_{0,1} \le 1 - r_1$. This fact is led similarly to the proof of Theorem 2. We employ (B) of Lemma 4. It turns out that the result of maximization applies (iii) of Lemma 3, that is to say, a type-012 strategy is optimal. □

5 Discussion

What one should note is that in the multislope ski-rental problem in general, the order of buying goods is fixed. That is to say, it is not allowed for the player to buy goods in an arbitrary order.

Indeed, in this paper the player is obliged to buy a ski wear first. It is interesting to assume a submodular set function that maps any subset of ski gear to the price, and to perform a competitive analysis allowing to buy goods in an arbitrary order.

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