Analysis of Lower Bounds for the Multislope Ski-Rental Problem

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Abstract

The multislope ski-rental problem is an extension of the classical ski-rental problem, where the player has several options of paying both of a per-time fee and an initial fee, in addition to pure renting and buying options. Damaschke gave a lower bound of 3.62 on the competitive ratio for the case where arbitrary number of options can be offered. In this paper we propose a scheme that for the number of options given as an input, provides a lower bound on the competitive ratio, by extending the method of Damaschke. This is the first to establish a lower bound for each of the 5-or-more-option cases, for example, a lower bound of 2.95 for the 5-option case, 3.08 for the 6-option case, and 3.18 for the 7-option case. Moreover, it turns out that our lower bounds for the 3- and 4-option cases respectively coincide with the known upper bounds. We therefore conjecture that our scheme in general derives a matching lower and upper bound.

1 Introduction

In the *ski-rental problem* the player is offered two options for getting his/her ski gear: either to *rent* ski gear by paying a fee each time of skiing, or to *buy* it. Once the player has bought ski gear, it is available for free forever. The objective is to minimize the total cost, under the setting that the player does not know how many times he/she is going skiing in the future. A strategy of such a player is to rent ski gear for the time being and then buy it. The performance of a strategy is measured by the *competitive ratio*. We say the competitive ratio of a strategy to be c if the player according to the strategy is charged at most c times the optimal offline cost, i.e., one with the number of times of skiing known in advance. A matching upper and lower bound on the competitive ratio is known to be 2 [KMRS88]. In other words, for any price setting, there exists a strategy with a competitive ratio of 2 and it cannot be improved.

The *multislope ski-rental problem* is an extension of the ski-rental problem [AIS08, LPSR08]. The player here has not only the pure rent and buy options, but also several options of paying *both of some per-time fee and some initial fee*, for example, to rent only a pair of skis and boots after having bought other gear like ski clothes. For this problem, an upper bound of 4 [BCN00] and a lower bound of 3.62 [Dam03] are known.

When it comes to application, however, the result for the multislope ski-rental problem seems less helpful, since each of these bounds becomes tight in the case where sufficiently many options are offered to the player. We should mention here that the multislope ski-rental problem can be thought as *Dynamic Power Management* [ISG03] on an electronic device equipped with multiple *energy-saving states*, such as a Windows computer equipped with Sleep, Stand By, and Hibernate states. The energy-saving states correspond to the options in the multislope ski-rental problem. The objective is to minimize the energy consumption during an idle time with its length unknown. In reality, it seems that the number of energy-saving states on a device is at most ten or so. Our aim is thus to seek a lower bound for such a realistic case.

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Table 1: Lower and upper bounds for the multislope ski-rental problem. Bold font indicates bounds established in this paper. "# states" means how many options are offered to the player.

# states	2	3	4
Lower bound	2 [KMRS88]	2.47 [FKF11]	2.75 [FKF11]
Upper bound	2 [KMRS88]	2.47 [FKF11]	2.75 [FKF11]

# states	5	6	7	8	9	10	general
Lower bound	**2.95**	**3.08**	**3.18**	**3.25**	**3.31**	**3.36**	3.62 [Dam03]
Upper bound	4 [BCN00]						

1.1 Our Contribution

In this paper we reveal lower bounds on the competitive ratio for the multiple ski-rental problem for the case where the number of options is specified. More precisely, we design a scheme that provides a lower bound with the number of options given as an input, extending the method of Damaschke [Dam03]. See Table 1. This work is the first to establish a lower bound for each of the 5-or-more-option cases. We show a lower bound of 2.95 for the 5-option case, 3.08 for the 6-option case, 3.18 for the 7-option case, and so on.

Besides, it turns out that our lower bounds for the 3- and 4-option cases, 2.47 and 2.75, exactly coincide with the matching upper and lower bounds identified in [FKF11], respectively. We therefore conjecture that also for the 5-or-more-option cases, the output of our scheme is equal or quite close to the matching lower and upper bound.

1.2 Related Work

The ski-rental problem was first introduced as an optimization model of snoopy caching by Karlin et al. [KMRS88], where a matching lower and upper bound of 2 was derived.

The start point of research on the multislope ski-rental problem can be found in the paper of Irani et al. in the context of Dynamic Power Management [ISG03]. They studied online strategies for switching multiple energy-saving states. Augustine et al. [AIS08] proposed an algorithm that for a given instance of the multislope ski-rental problem, outputs the best possible strategy and its competitive ratio. The best lower bound so far of 3.62 was established by Damaschke [Dam03], on which our contribution is mainly based. On the other hand, the best upper bound so far of 4 was shown by Bejerano et al. [BCN00]. These two works preceded the introduction of the multislope ski-rental problem, since they are a corollary from the results on the *online investment problem* which can be regarded as a special case of the multislope ski-rental problem, in which the player is obliged to buy new gear every time changing options. Fujiwara et al. [FKF11] gave a matching lower and upper bound for each of the 3- and 4-option cases: 2.47 and 2.75, respectively.

To establish a lower bound is, in other words, to reveal the hardest instance for the player. For the multislope ski-rental problem the easiest instance had also been non-trivial. Fujiwara et al. [FKF11] showed the easiest instance and the best possible competitive ratio for it.

1.3 Note on Rounding

Throughout this paper numerical rounding is all done to the nearest value. It would be a conventional manner that a lower bound is rounded down while an upper bound is rounded up. The reason why we nevertheless insist on rounding to the nearest is because there appear some matching lower and upper bounds, such as those in Table 1. Refer to Table 2 for the values with more precision.

2 Problem Statement

2.1 Instance

An instance of the $(k+1)$-*slope ski-rental problem* consists of $k+1$ *states*, each of which stands for a way to get the player's ski gear. We collectively refer to the $(k+1)$-slope ski-rental problem for all k as the *multislope ski-rental problem.* Although we have called a state an *option* in Section 1 for ease of understanding, we will hereafter use the word "state". This is referred to as a *slope* in some literature. State 0 and state k are *to rent* and *to buy*, respectively. States $1, \ldots, k-1$ correspond to the options that the player pays *both of some per-time fee and some initial fee*, for example, to rent only a pair of skis and boots after having bought other gear such as ski clothes. Let r_i and $b_{i,j}(\geq 0)$ denote the per-time fee of state i and the initial fee for transitioning from state i to j, respectively. In this paper we impose the following natural constraints:

$$r_0 = 1, r_k = 0, b_{0,k} = 1 \tag{1}$$

$$r_i > r_j \text{ for } 0 \leq i < j \leq k, \tag{2}$$

$$b_{l,j} - b_{l,i} \leq b_{i,j} \leq b_{l,j} \text{ for } 0 \leq l < i < j \leq k. \tag{3}$$

(1) normalizes so that per-time and initial fees are all scaled down to between zero and one. This normalization may look somewhat strange, but it makes sense; the number of times of skiing will also be scaled soon. (2) says that the states are ordered so that the per-time fee decreases. The left inequality in (3) is a constraint that a direct transition from state l to j is equal to or cheaper than one shortly stopping another state i. The right inequality in (3) says that a transition from state i to j is no cheaper than one from state $l < i$. An instance is thus represented by a pair of such vectors $(\boldsymbol{r}, \boldsymbol{b})$.

For example, a store may offer the following options: Rent everything for \$50 per day (state 0), buy everything for \$500 (state 2), or rent just skis and boots for \$30 per day with buying other gear for \$100 (state 1). Also, the store may allow you to change state 1 to 2 by charging \$450. Then, we may formulate as $(r_0, r_1, r_2) = (50/50, 30/50, 0) = (1, 0.6, 0)$ and $(b_{0,1}, b_{0,2}, b_{1,2}) = (100/500, 500/500, 450/500) = (0.2, 1, 0.9)$.

2.2 Strategy and Competitiveness

Without loss of generality, we assume the number of times of skiing is a real number $t \geq 0$. We sometimes identify t with the time during which the player repeatedly goes skiing. At each time instant, the player at state i, who is paying r_i per time unit, either (a) transitions to a different state j by paying $b_{i,j}$ or (b) keeps staying at state i. In this setting a *strategy* of the player is described by a vector $\boldsymbol{x}$ with $k+1$ entries. Each entry x_i indicates the time when the player transitions to state i from state $i-1$. The sequence of the entries is assumed to be non-decreasing, since we consider only such instances for which the player cannot save cost by backward transition, due to the constraint (2). Since state 0 incurs no initial fee, we can assume that $x_0 = 0$, i.e., the player always starts from state 0. The player may transition from state i

directly to j by skipping the states between. For such a transition, we set $x_{i+1} = \cdots = x_{j-1} = x_j$ and define a relation of $i \prec j$. By time t such that $x_i \leq t < x_{i+1}$, the player according to strategy $\boldsymbol{x}$ will have paid a cost of

$$ON(\boldsymbol{x}, t) := r_i\,(t - x_i) + \sum_{l=0}^{i-1} r_l\,(x_{l+1} - x_l) + \sum_{0 \leq l \prec m \leq i} b_{l,m}.$$

The first and second terms are the sum of per-time fees for the states chosen so far, and the third is the sum of initial fees. On the other hand, the *optimal offline player* behaves optimally with t known. Due to the constraint (3), the optimal offline player will choose the best state for him/her at the beginning and then keep staying there. Therefore, the cost incurred by time t is written as

$$OPT(t) := \min_{0 \leq j \leq k} (r_j t + b_{0,j}).$$

We measure the performance of a strategy using the *competitive ratio*, which is a standard measure in online optimization [BE98]. We say that strategy $\boldsymbol{x}$ has a competitive ratio of c, if

$$ON(\boldsymbol{x}, t) \leq c \cdot OPT(t)$$

holds for all $t \geq 0$.

2.3 Investment Instance

In the rest of this paper we consider the subset of instances that satisfies an additional constraint

$$b_{i,j} = b_{0,j} \text{ for } 0 < i < j \leq k,$$

which means in reality that when the player transitions to a state, his/her own gear cannot be reused and therefore he/she is obliged to buy new gear from scratch. In [FKF11] such an instance is referred to as an *investment instance*, which corresponds to *online capital investment* [Dam03]. Note that in the above numerical example, we can have an investment instance if we change $b_{1,2}$ into one.

In the following two sections we will deal with only investment instances and therefore write $b_{i,j}$ simply as b_j. It is demonstrated how to determine $\boldsymbol{b} = (b_1, \ldots, b_k)$ so that a lower bound is established. We add also that there will no longer appear any explicit strategy $\boldsymbol{x}$; our discussion is based on a known lemma (Lemma 1) which bounds the competitive ratio of arbitrary strategies.

3 General Lower Bound

Damaschke derived a lower bound of $\frac{5+\sqrt{5}}{2} \approx 3.62$ for the general case, that is to say, for the case where arbitrary number of states can be offered [Dam03]. We here review the derivation of the result in such a way that we can later extend it to the case where the number of options is fixed. We first extract the construction of $\boldsymbol{b}$ with some modification. Although Damaschke constructed $\boldsymbol{r}$ and $\boldsymbol{b}$ through a single procedure, we here discuss them separately.

Procedure 1. *(for determining $\boldsymbol{b}$, [Dam03]) Given $c \geq 2$ as a parameter, starting with*

$$q_1(c) = \frac{c}{c-1}, \tag{4}$$

determine sequence $\{q\}$ according to

$$q_{i+1}(c) = \frac{c^3 - c^2 + cq_i(c)}{c^3 - c^2 - (c-1)^2 q_i(c)}. \tag{5}$$

Let $m(c)$ be the largest integer such that for all $1 \le i \le m(c)$, $q_i(c) \le c$ holds. Next, starting with $b_{m(c)+1} = 1$, determine sequence $\{b\}$ in a reverse order by

$$b_i = \frac{b_{i+1}}{c - \frac{c-1}{c} q_i}$$

for $1 \le i \le m(c)$. Case (i): if $q_{m(c)}(c)$ is exactly equal to c, output $\boldsymbol{b} = (b_1, \ldots, b_{m(c)})$. (Note that due to (5), $b_{m(c)} = b_{m(c)+1}$ holds.) Case (ii): otherwise, i.e., if $q_{m(c)}(c) < c$, output $\boldsymbol{b} = (b_1, \ldots, b_{m(c)+1})$.

It is observed that $q_{i+1}(c)$ is defined for every $q_i(c)$ with $1 \le i \le m(c)$, since noting $q_i(c) \le c$, the denominator of the right hand side of (5) is

$$\begin{aligned} c^3 - c^2 - (c-1)^2 q_i(c) &= (c-1)^2 (c + 1 + \frac{1}{c-1} - q_i(c)) \\ &\ge (c-1)^2 (1 + \frac{1}{c-1}) \\ &> 0. \end{aligned}$$

With some proper $\boldsymbol{r}$, an $(m(c)+1)$-state instance is generated from Case (i), and an $(m(c)+2)$-state instance from Case (ii). In the later discussion, an instance from Case (i) plays a significant role. We here give a numerical example of $c = 2.5$. We first have $q_1 = 1.66667$, $q_2 = 2.40741$. Since the next term $q_3 = 3.88889$ is bigger than 2.5, we know $m(c) = 2$. Then, corresponding to Case (ii), $\boldsymbol{b}$ is determined as $(b_1, b_2, b_3) = (0.631579, 0.947368, 1)$.

In Procedure 1 above, we have made two modifications to the original construction, which do not affect the argument of competitiveness. One is that the values in $\boldsymbol{b}$ are scaled down between zero and one. The other is that we have clarified the condition of stopping the generation of sequence $\{q\}$.

Lemma 1. *([Dam03]) For $c \ge 2$, determine $\boldsymbol{b}$ according to Procedure 1. Then, for all $\varepsilon > 0$, there exists $\boldsymbol{r}$ such that any strategy for instance $(\boldsymbol{r}, \boldsymbol{b})$ has a competitive ratio of at least $\min(c - \varepsilon, q_{m(c)})$.*

The purpose of Procedure 1 is to obtain this lemma. According to the original description [Dam03], state $i+1$ is inductively constructed from states i and $i-1$ so that if the strategy skips state i, then the competitive ratio becomes above $c - \varepsilon$. Consequently, any strategy with one or more skips of states has a competitive ratio of at least $c - \varepsilon$, while any strategy with no skip has a competitive ratio of at least $q_{m(c)}$. The values of $r_1, r_2, \ldots, r_{k-1}$ are chosen sufficiently small so that they form a decreasing sequence.

For example, applying Lemma 1 to the above example, we know that any strategy for that instance has a competitive ratio larger than 2.40741.

The paper [Dam03] then states that for any $c = \frac{5+\sqrt{5}}{2} - \delta$ with $\delta > 0$, there exists i such that $q_i(c) > c$, which leads to the conclusion below.

Theorem 1. *([Dam03]) Any strategy for the multislope ski-rental problem has a competitive ratio of at least $\frac{5+\sqrt{5}}{2} (\approx 3.62)$.*

4 Lower Bound for the Case where the Number of States is Fixed

4.1 Our Scheme

In this section we obtain a lower bound for the case where the number of states is explicitly specified. The general lower bound derived in Section 3 involves an instance with sufficiently many states; $m(c)$ grows as c approaches $\frac{5+\sqrt{5}}{2}$. Our aim is to construct a $(k+1)$-state instance so that $q_k(c)$ is equal to c, exploiting Procedure 1 and Lemma 1.

The key is the choice of c. A $(k+1)$-state instance can simulate an instance with k or fewer states by making redundant states. Besides, it is known that a lower bound for 2-state (i.e., classical) ski-rental problem is 2 [KMRS88]. Therefore, the range of candidates for c is between 2 and $\frac{5+\sqrt{5}}{2}$.

It is yet unclear how many states Procedure 1 outputs for given c. One should take good care in handling the sequence $\{q\}$. If one does not terminate the generation as we do there, the sequence is in general not either increasing or decreasing, and may even include a negative term.

We propose a simple scheme for obtaining a lower bound: *For given k, solve formally the equation $c = q_k(c)$, derived from (4) and (5) without plugging a value into c, and take the largest root among real roots which lie between 2 and $\frac{5+\sqrt{5}}{2}$.* The following arguments will guarantee that the largest root in fact leads to a $(k+1)$-state instance and is consequently a lower bound for the $(k+1)$-slope ski-rental problem.

We begin with analyzing the behavior of each $q_i(c)$ as a function of c. Although the general term of $q_i(c)$ can be derived in a closed form with the generating function method, we do not carry it out here.

Lemma 2. *Run Procedure 1 with c_0 such that $2 < c_0 < \frac{5+\sqrt{5}}{2}$. For $1 \le i \le m(c_0)+1$, see each $q_i(c)$ as a function $q_i : c \mapsto q_i(c)$. Then, q_i is continuous and monotonically decreasing on the interval $[c_0, \frac{5+\sqrt{5}}{2})$.*

Proof. We prove the lemma by induction. In addition to the statement of the lemma, we show the differentiability on $[c_0, \frac{5+\sqrt{5}}{2})$ as well. It is easy to confirm that $q_1(c) = \frac{c}{c-1}$ is continuous, differentiable, and monotonically decreasing on $[c_0, \frac{5+\sqrt{5}}{2})$.

Suppose that for some i with $1 \le i \le m(c_0)$, q_i is continuous and differentiable, and $\frac{dq_i(c)}{dc} < 0$ holds on $[c_0, \frac{5+\sqrt{5}}{2})$. From (5), we immediately know that q_{i+1} is continuous and differentiable on $c \in [c_0, \frac{5+\sqrt{5}}{2})$. Denoting $q_i(c)$ simply as q, we have

$$\frac{dq_{i+1}(c)}{dc} = \frac{1}{(c-1)^2(c^2+q-cq)^2} \cdot \left\{-q(c^4-2c^3+4c^2-2c-(c^2-1)q)+c^2(c-1)(c^2-c+1)\frac{dq}{dc}\right\} \tag{6}$$

See the denominator. By the assumption it follows that $c \ge c_0 \ge q_i(c_0) \ge q$. Then, since

$$\begin{aligned} c^2+q-cq &= (c-1)(c+1+\frac{1}{c-1}-q) \\ &\ge (c-1)(1+\frac{1}{c-1}) \\ &> 0, \end{aligned}$$

$\frac{dq_{i+1}(c)}{dc}$ is always defined. The proof is done if we show that on $[c_0, \frac{5+\sqrt{5}}{2})$, $\frac{dq}{dc} < 0$ implies $\frac{dq_{i+1}(c)}{dc} < 0$. Let us see the numerator of (6). We easily have for $c \geq 2$,

$$c^2(c-1)(c^2-c+1) > 0.$$

We finally look into the function

$$h(c,q) := c^4 - 2c^3 + 4c^2 - 2c - (c^2-1)q.$$

Apparently, this function decreases as q grows. Thus, for $q \leq c_0 < \frac{5+\sqrt{5}}{2} < 4$, $h(c,q) > h(c,4) = (c-2)(c^3-2) \geq 0$. Therefore, $\frac{dq_{i+1}(c)}{dc} < 0$ if $\frac{dq}{dc} < 0$. □

Intuitively, the following Lemma 4 claims that the number of states can always be increased by choosing larger c. Before presenting it, we give a helper lemma, which is a statement found in [Dam03].

Lemma 3. *([Dam03]) For all $i \geq 1$, $q_i(\frac{5+\sqrt{5}}{2}) < \sqrt{5}$.*

Lemma 4. *The following two statements hold true. (A) For all $2 < c \leq c' < \frac{5+\sqrt{5}}{2}$, $m(c') \geq m(c)$. (B) For all $2 < c < \frac{5+\sqrt{5}}{2}$, there exists c' such that $c < c' < \frac{5+\sqrt{5}}{2}$, $m(c') = m(c)+1$, and $q_{m(c')}(c') = c'$.*

Proof. (A) By the definition of m, $q_i(c) \leq c$ holds for all $1 \leq i \leq m(c)$. Lemma 2 implies that for any $c \leq c' < \frac{5+\sqrt{5}}{2}$, $q_i(c') \leq q_i(c)$ for each i. We thus have for all $1 \leq i \leq m(c)$, $q_i(c') \leq q_i(c) \leq c \leq c'$. Going back to the definition of m, it is concluded that $m(c') \geq m(c)$.

(B) See the functions $x \mapsto q_{m(c)+1}(x)$ and $x \mapsto x$ on the interval $[c, \frac{5+\sqrt{5}}{2})$. By the definition of m, it holds that $q_{m(c)+1}(c) > c$. Lemma 3 guarantees $q_{m(c)+1}(\frac{5+\sqrt{5}}{2}) < \sqrt{5} < \frac{5+\sqrt{5}}{2}$. Lemma 2 says that the function $x \mapsto q_{m(c)+1}(x)$ is continuous and monotonically decreasing on $[c, \frac{5+\sqrt{5}}{2})$. Therefore, the two functions must have an intersection point $c' \in (c, \frac{5+\sqrt{5}}{2})$ such that $q_{m(c)+1}(c') = c'$.

The rest is to prove $m(c') = m(c)+1$ for such c'. Note that if this is done then $q_{m(c')}(c') = c'$ immediately follows. Similarly to (A), we know that for all $1 \leq i \leq m(c)$, $q_i(c') \leq q_i(c) \leq c \leq c'$. In addition, $q_{m(c)+1}(c') = c'$ also holds. Therefore, $m(c') \geq m(c)+1$. On the other hand, applying (5), we have

$$\begin{aligned} q_{m(c)+2}(c') &= \frac{c'^3 - c'^2 + c' q_{m(c)+1}(c')}{c'^3 - c'^2 - (c'-1)^2 q_{m(c)+1}(c')} \\ &= \frac{c'^3 - c'^2 + c'^2}{c'^3 - c'^2 - c'(c'-1)^2} \\ &= c' + 1 + \frac{1}{c'-1} \\ &> c', \end{aligned} \tag{7}$$

which implies that $m(c') \leq m(c)+1$. Hence, $m(c') = m(c)+1$. □

The next lemma is the heart of our scheme.

Lemma 5. *For given $k \geq 2$, solve the equation $q_k(c) = c$ formally. Let $\overline{c}$ be the maximum of its real roots which lie between 2 and $\frac{5+\sqrt{5}}{2}$. Procedure 1 with $\overline{c}$ outputs **b** with k entries.*

Table 2: Our lower bound and the equation that has a root of that value. Each equation is displayed in a reduced form by eliminating irrelevant factors.

# states	Lower bound	Equation
3	2.46557	$c^3 - 4c^2 + 5c - 3 = 0$
4	2.75488	$c^3 - 5c^2 + 8c - 5 = 0$
5	2.94789	$c^5 - 7c^4 + 19c^3 - 27c^2 + 21c - 8 = 0$
6	3.08302	$c^5 - 8c^4 + 25c^3 - 40c^2 + 34c - 13 = 0$
7	3.18123	$c^7 - 10c^6 + 42c^5 - 99c^4 + 145c^3 - 135c^2 + 76c - 21 = 0$
8	3.25479	$c^7 - 11c^6 + 51c^5 - 132c^4 + 210c^3 - 209c^2 + 123c - 34 = 0$
9	3.31128	$c^9 - 13c^8 + 74c^7 - 246c^6 + 534c^5 - 795c^4 + 822c^3 - 577c^2 + 254c - 55 = 0$
10	3.35558	$c^9 - 14c^8 + 86c^7 - 308c^6 + 717c^5 - 1137c^4 + 1241c^3 - 909c^2 + 411c - 89 = 0$

Table 3: $\boldsymbol{b}$ of the instance that achieves our lower bound.

# states	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
3	0.682328	1							
4	0.398445	0.699223	1						
5	0.216735	0.422177	0.711088	1					
6	0.113682	0.236803	0.43869	0.719345	1				
7	0.0584374	0.127465	0.25124	0.450502	0.725251	1			
8	0.0296859	0.0669354	0.137759	0.261927	0.459209	0.729605	1		
9	0.0149719	0.0346041	0.0735019	0.145631	0.270046	0.465803	0.732901	1	
10	0.00751682	0.0177065	0.0385179	0.0786695	0.151777	0.276354	0.470914	0.735457	1

Proof. Let M be the set of real roots of the equation $q_k(c) = c$ which are in $(2, \frac{5+\sqrt{5}}{2})$.

(I) First we show that there exists $c \in M$ such that $m(c) = k$. Let us generate sequence $\{q\}$ with some small c, say $c_1 = 11/5 = 2.2$. We have $q_1(c_1) = 11/6 < 11/5$ and $q_2(c_1) = 671/216 > 11/5$. Hence, $m(c_1) = 1$. Repeatedly applying (B) of Lemma 4 to this for $k-1$ times, we find c such that $m(c) = k$ and $q_{m(c)}(c) = q_k(c) = c$. The found c surely belongs to M.

(II) Next we claim that $m(c) \leq k$ for all $c \in M$. By the definition of M, any $c \in M$ satisfies $q_k(c) = c$. What should be noted is that some $q_k(c)$ may not be the last element of the sequence $\{q\}$ defined by Procedure 1. In other words, for some $c \in M$, the generation of the sequence may stop earlier. Even for such c, similarly to (7), we derive $q_{k+1}(c) > c$, which implies that $m(c) \leq k$ by the definition of m.

(III) By (A) of Lemma 4 and the above (II), it follows that for all $c \in M$, $m(c) \leq m(\bar{c}) \leq k$. Together with (I), we conclude $m(\bar{c}) = k$. Since $q_k(\bar{c}) = \bar{c}$, Case (i) of Procedure 1 applies. Therefore, Procedure 1 outputs $\boldsymbol{b}$ with k entries. □

Our main theorem follows immediately from Lemmas 1 and 5.

Theorem 2. *Any strategy for the $(k+1)$-slope ski-rental problem has a competitive ratio of at least the maximum of real roots of the equation $q_k(c) = c$ which lie between 2 and $\frac{5+\sqrt{5}}{2}$.*

Table 2 shows our lower bound for each $2 \leq k \leq 9$ and the equation that has a root of that value. See Table 3 for numerical values of $\boldsymbol{b}$. The reason why only up to the 10-state case is presented here is simply because of space limitation. See Figure 1 for a plot of our lower bound for $2 \leq k \leq 29$.

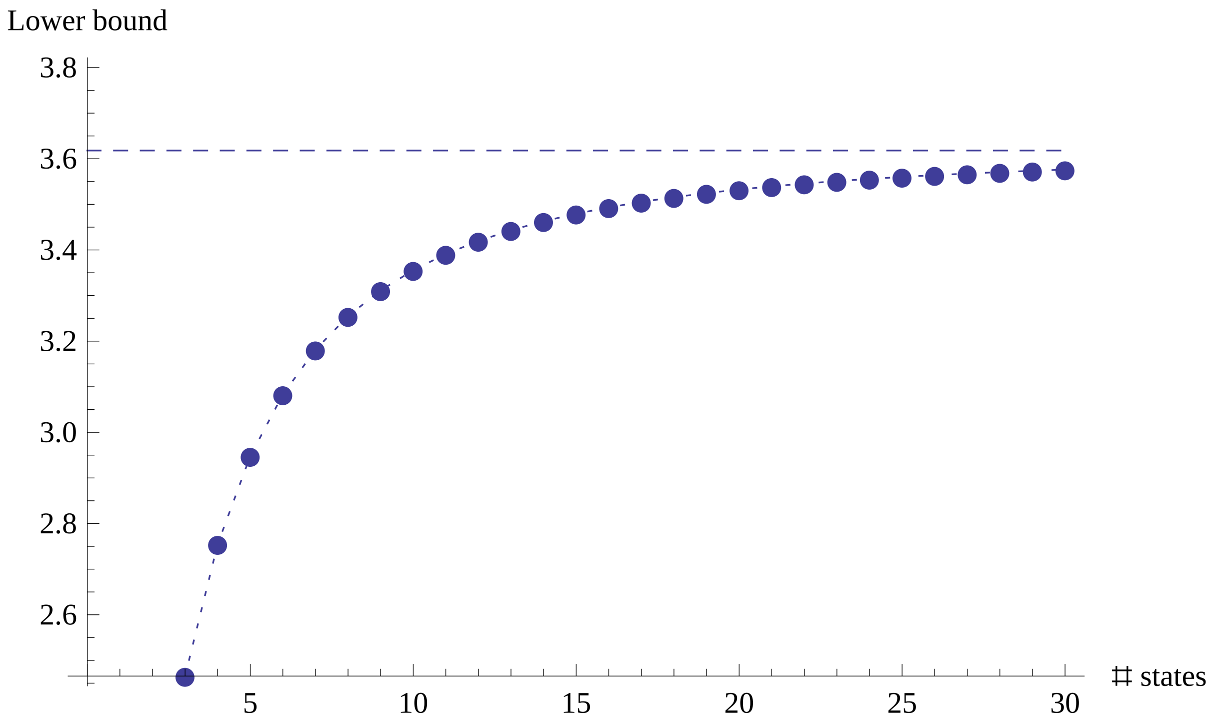

Figure 1: Our lower bound for $2 \leq k \leq 29$. The dashed line is at $\frac{5+\sqrt{5}}{2} \approx 3.62$.

If one is interested only in the numerical value of the lower bound, one needs not even derive the equation $q_k(c) = c$ in an explicit form. Since for any value of c, $q_k(c)$ can be calculated by repeatedly applying (5), a binary search on c leads to a numerical root of $q_k(c) = c$.

4.2 Matching Lower and Upper Bound for the 3- and 4-Slope Problems

For each of the 3- and 4-state cases, the matching lower and upper bound has been already established as below [FKF11]. One will quickly notice that for each case, our lower bound matches the matching bound, in the sense that the equation of c is the same (see Table 2). From this observation, it is likely that also for the 5-or-more-state cases, our lower bound is equal or fairly close to the matching lower and upper bound.

Theorem 3. *([FKF11]) Any strategy for the* 3*-slope ski-rental problem has a competitive ratio of at least the real root of the equation* $c^3 - 4c^2 + 5c - 3 = 0$*, which is approximately* 2.47.

Theorem 4. *([FKF11]) Any strategy for the* 4*-slope ski-rental problem has a competitive ratio of at least the real root of the equation* $c^3 - 5c^2 + 8c - 5 = 0$*, which is approximately* 2.75.

5 Concluding Remarks

For the case where arbitrary number of states are allowed, there remains a gap between the lower and upper bounds of 3.62 and 4. Damaschke in his paper [Dam03] conjectured that the matching lower and upper bound is 4. Our conjecture is, in contrast, that the matching bound is 3.62. In other words, the lower bound derived from Damaschke's method is the best possible. In Section 4.2, we have already provided an evidence for the 3- and 4-state cases that Damaschke's method with our extension establishes the matching bound.

A future work is to design a simpler scheme, or hopefully to express the lower bound explicitly with k. Another interesting work is to improve the upper bound of 4.

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